Constructive Logic (15-317), Fall 2014
Assignment 2: Proof Terms and Substitution

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Out: Thursday, September 11, 2014
Due: Thursday, September 18, 2014 (before class)

In this assignment, you will write proof terms and demonstrate your understanding of substitution.

Homework submission will be done differently starting with this assignment.

The written portion of your work (Sections 2, 3, and 4) should be submitted electronically before the beginning of class. Please convert your homework to a PDF file titled `hw02.pdf`, and put the file in:

```
/afs/andrew/course/15/317/submit/<your andrew id>
```

If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand and scan them.

The Tutch portion of your work (Section 1) should be submitted electronically using the command

```
$ /afs/andrew/course/15/317/bin/submit -r hw02 <files...>
```

from any Andrew server. You may check the status of your submission by running the command

```
$ /afs/andrew/course/15/317/bin/status hw02
```

If you have trouble running either of these commands, email Joe or Evan.

1 Tutch Proofs (15 points)

To give a proof term in Tutch, declare it with `term` rather than `proof`:

```
term andComm : A & B => B & A =
  fn u => (snd u, fst u);
```

For more examples, see Chapter 4 of the *Tutch User's Guide*. The proof terms are very similar to the ones given in lecture (even more similar to ML) and are summarized in Section A.2.1 of the *Guide*.

Task 1 (15 pts)**.** Prove the following theorems using Tutch:

```
term mpt: ~(A & B) => A => ~B;
term disjImp: (A => C) => (B => C) => (A | B => C);
term branch: (A & B => C) => (A | C) => (B | C) => C;
term flip : (A => ~B) => (B => ~A);
term demorgan : ~(A | B) => ~A & ~B;
```

2 Reduction (6 points)

Recall the definition of proof term reduction given in lecture. Although we have not proven this, every valid proof term can only be reduced a finite number of times. Here is an example:

$$\begin{aligned} & (\lambda u{:}A \wedge B.\, \pi_1 u)\, \langle a, b\rangle \\ \Longrightarrow_R\;& \pi_1\langle a, b\rangle \\ \Longrightarrow_R\;& a \end{aligned}$$

Task 2 (6 points)**.** Show a step-by-step reduction of the following terms, until you reach a term that cannot be reduced further. There may be more than one correct sequence of reductions. *Warning*: be careful to avoid variable capture when substituting!

1. $(\lambda y{:}A.\, \lambda f{:}A \supset B.\, f\ y)\ (f\ y)$
2. $\lambda u{:}A \wedge B.\, \big(\lambda v{:}B \wedge A.\, \langle \pi_2 v, \pi_1 v\rangle\big)\ \big((\lambda u{:}A \wedge B.\, \langle \pi_2 u, \pi_1 u\rangle)\ u\big)$
3. $\big(\lambda x{:}A \vee B.\, \mathsf{case}(x, x.\lambda y{:}A.\, \langle x, y\rangle, x.\lambda y{:}A.\, \langle y, y\rangle)\big)\ (\mathsf{in}_1\ y)$

3 Verification and Uses (12 points)

In this set of tasks, we will investigate how replacing the *true* judgment with the $\uparrow$ and $\downarrow$ judgments affects the structure of proofs. Assume A and B are atomic propositions.

Task 3 (3 points)**.** Give *two* different natural deduction proofs (or proof terms) of $A \supset A$ *true*. How many natural deduction proofs of this judgment exist? Explain.

Task 4 (3 points)**.** Give a proof of $A \supset A \uparrow$. How many proofs of this judgment exist? Explain.

Task 5 (3 points)**.** Give *two* different natural deduction proofs of $(A \vee B) \supset (B \vee A)$ *true*. How many natural deduction proofs of this judgment exist? Explain.

Task 6 (3 points)**.** Give a proof of $(A \vee B) \supset (B \vee A) \uparrow$. How many proofs of this judgment exist? Explain.

4 Follow your ♡ (7 points)

4.1 Uses and verifications

Recall the ♡ connective defined in Homework 1:

$$\dfrac{\begin{matrix}[A\ \mathit{true}] \\ \vdots \\ C\ \mathit{true}\end{matrix} \quad \begin{matrix}[B\ \mathit{true}] \\ \vdots \\ C\ \mathit{true}\end{matrix}}{\heartsuit(A,B,C)\ \mathit{true}}\ \heartsuit I \qquad \dfrac{\heartsuit(A,B,C)\ \mathit{true} \quad A\ \mathit{true}}{C\ \mathit{true}}\ \heartsuit E_1 \qquad \dfrac{\heartsuit(A,B,C)\ \mathit{true} \quad B\ \mathit{true}}{C\ \mathit{true}}\ \heartsuit E_2$$

Task 7 (3 points)**.** Give use and verification rules corresponding to these introduction and elimination rules. There may be more than one right answer.

4.2 Proof terms

Consider the following proof term assignment for the ♡ connective:

$$\dfrac{\begin{matrix}[x : A] \\ \vdots \\ M : C\end{matrix} \quad \begin{matrix}[y : B] \\ \vdots \\ N : C\end{matrix}}{\langle\!\langle x.M, y.N \rangle\!\rangle_C : \heartsuit(A,B,C)}\ \heartsuit I \qquad \dfrac{M : \heartsuit(A,B,C) \quad N : A}{p_1(M,N) : C}\ \heartsuit E_1 \qquad \dfrac{M : \heartsuit(A,B,C) \quad O : B}{p_2(M,O) : C}\ \heartsuit E_2$$

Task 8 (4 points)**.** Show all the local reduction(s) and expansion(s) for these rules in proof term notation. Be sure to indicate which are reductions and which are expansions. You do not need to include derivations for the proof terms.