Midterm II

15-317: Constructive Logic

November 6, 2014

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Instructions

- This exam is closed book and closed Internet. A two-sided sheet of handwritten notes is permitted.
- There are three problems. Not all problems are the same size or difficulty, so it may help to read through the whole exam first. You have ninety minutes to complete the exam.
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Good luck!

	Problem 1	Problem 2	Problem 3	**Total**
Score				
Max	30	40	30	100

1 Invertibility

Problem 1: Below are three rules governing a connective $\spadesuit(A, B, C)$ in natural deduction. Which of the rules are invertible, and which are non-invertible? Explain.

$$\dfrac{\begin{array}{cc}\begin{array}{c}A\ \mathsf{true}\\ \vdots\\ B\ \mathsf{true}\end{array} & \begin{array}{c}A\ \mathsf{true}\\ \vdots\\ C\ \mathsf{true}\end{array}\end{array}}{\spadesuit(A, B, C)\ \mathsf{true}}\ \spadesuit I \qquad \dfrac{\spadesuit(A, B, C)\ \mathsf{true} \quad A\ \mathsf{true}}{B\ \mathsf{true}}\ \spadesuit E1 \qquad \dfrac{\spadesuit(A, B, C)\ \mathsf{true} \quad A\ \mathsf{true}}{C\ \mathsf{true}}\ \spadesuit E2$$

The rule $\spadesuit I$ is invertible, because $\spadesuit E1$ and $\spadesuit E2$ can be used to recover its premises from its conclusion.

The rules $\spadesuit E1$ and $\spadesuit E2$ are not invertible, because B or C can certainly be true without $\spadesuit(A, B, C)$ and A being true. For example, we could have $A = F$ and $B = T$.

Problem 2: Below are three rules governing a connective $\clubsuit(A, B, C)$ in sequent calculus. Which of the rules are invertible, and which are non-invertible? Explain. (You may assume that cut admissibility and identity expansion hold.)

$$\frac{\Delta \longrightarrow A \quad \Delta \longrightarrow B}{\Delta \longrightarrow \clubsuit(A, B, C)}\ \clubsuit R1 \qquad \frac{\Delta \longrightarrow A \quad \Delta \longrightarrow C}{\Delta \longrightarrow \clubsuit(A, B, C)}\ \clubsuit R2$$

$$\frac{\Delta, A, B \longrightarrow D \quad \Delta, A, C \longrightarrow D}{\Delta, \clubsuit(A, B, C) \longrightarrow D}\ \clubsuit L$$

The rule $\clubsuit L$ is invertible, because $\clubsuit R1$ and $\clubsuit R2$ together with cut admissibility and identity expansion can be used to recover its premises. For the first premise, we can show $\Delta, A, B \longrightarrow A$ and $\Delta, A, B \longrightarrow B$ by identity expansion, so $\Delta, A, B \longrightarrow \clubsuit(A, B, C)$ by $\clubsuit R1$, so $\Delta, A, B \longrightarrow D$ by cut admissibility.

The rules $\clubsuit R1$ and $\clubsuit R2$ are not invertible, because $\Delta \longrightarrow \clubsuit(A, B, C)$ can be derivable without $\Delta \longrightarrow B$ and $\Delta \longrightarrow C$ being derivable. For example, we could have $A = C = T$ and $B = F$.

Problem 3: Explain the role that invertibility plays in automated theorem proving. Use complete sentences.

We can use invertibility to reduce the size of the search space in automated theorem proving. Applying an invertible rule never changes the provability of the outstanding goals, so an invertible rule can be applied eagerly without needing to record a backtracking point.

2 Logic Programming

Suppose we are given a collection of rules of the form `edge`$(from, to, cost)$ that define a directed acyclic graph with weighted edges. All costs are greater than zero. For example:

```
edge(a, b, 1).
edge(a, c, 1).
edge(b, d, 2).
edge(c, d, 2).
```

defines a graph with four nodes, in which one can move from `a` to `b` or `c` at a cost of 1, and from `b` or `c` to `d` at a cost of 2.

Problem 1: In Prolog, implement a predicate `path`(V, W, N) that holds when there exists a path (consisting of at least one edge) from V to W with a total cost of **exactly** N. Use Prolog's built-in arithmetic for cost arithmetic. Do not use cut.

```
path(V, W, N)
   :-
   edge(V, W, N).

path(V, W, N)
   :-
   edge(V, X, M),
   path(X, W, P),
   N is M+P.
```

Problem 2: Suppose the graph **may** contain cycles, all costs remain greater than zero, and `path` is always called with a ground integer as its third argument. Re-implement `path` so that it never enters an infinite loop. Use Prolog's built-in arithmetic for cost arithmetic. Do not use cut.

```
path(V, W, N)
   :-
   edge(V, W, N).

path(V, W, N)
   :-
   N > 0,
   edge(V, X, M),
   P is N-M,
   path(X, W, P).
```

3 Twelf

Below is Twelf code for natural numbers and addition, with mode, termination, and coverage checking.

```
nat : type.
z : nat.
s : nat -> nat.

plus : nat -> nat -> nat -> type.
%mode plus +M +N -P.

plus/z : plus z N N.

plus/s : plus (s M) N (s P)
          <- plus M N P.

%worlds () (plus _ _ _).
%total M (plus M _ _).
```

Problem 1: Below is Twelf code for multiplication. Either mode checking, termination checking, or coverage checking fails. Explain which one fails, and why. Then correct the program. Do not change the mode declaration.

```
mult : nat -> nat -> nat -> type.
%mode mult +M +N -P.

mult/z : mult z N z.

mult/s : mult (s M) N Q
          <- mult M N P
          <- plus P N Q.

%worlds () (mult _ _ _).
%total N (mult _ N _).
```

Termination checking fails because the indicated induction argument (the second argument) does not decrease in recursive calls. The final line should be replaced by:

```
%total N (mult N _ _).
```

Problem 2: Below is Twelf code for exponentiation. Either mode checking, termination checking, or coverage checking fails. Explain which one fails, and why. Then correct the program. Do not change the mode declaration. You may assume that the implementation of `mult` is correct.

```
exp : nat -> nat -> nat -> type.
%mode exp +M +N -P.

exp/z : exp N z (s z).

exp/s : exp M (s N) Q
         <- mult P M Q
         <- exp M N P.

%worlds () (exp _ _ _).
%total N (exp _ N _).
```

Mode checking fails because `P` is not ground in the call to `mult`. The order of the two subgoals in `exp/s` should be reversed.

Problem 3: Below is Twelf code for factorial. Either mode checking, termination checking, or coverage checking fails. Explain which one fails, and why. Then correct the program. Do not change the mode declaration. You may assume that the implementation of `mult` is correct.

```
fact : nat -> nat -> type.
%mode fact +M -N.

fact/1 : fact (s z) (s z).

fact/s : fact (s N) Q
          <- fact N P
          <- mult (s N) P Q.

%worlds () (fact _ _).
%total N (fact N _).
```

Coverage checking fails because the case where the input is `z` is not covered. The first rule should be replaced by:

```
fact/z : fact z (s z).
```