Midterm I

15-317: Constructive Logic

October 2, 2014

Name: Andrew ID:

Instructions

- This exam is closed book and closed Internet. A two-sided sheet of handwritten notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are three problems. Not all problems are the same size or difficulty, so it may help to read through the whole exam first. You have ninety minutes to complete the exam.
- When writing derivations, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^x$).
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Good luck!

	Problem 1	Problem 2	Problem 3	**Total**
Score				
Max	30	40	30	100

1 Constructive Derivations

A curious result in classical logic is that we can show, for any propositions P and Q, that $(P \supset Q) \vee (Q \supset P)$. This holds — in classical logic — because P is either true or false. If P is true, than anything implies P. If P is false, then P implies anything.

Problem 1: Give a derivation, in verifications and uses, of the judgement:

$$(P \vee \neg P) \supset ((P \supset Q) \vee (Q \supset P)) \uparrow$$

Assume P and Q are atomic.

$$\dfrac{\dfrac{[P \vee \neg P \downarrow]_x \quad \dfrac{\dfrac{\dfrac{[P \downarrow]_y}{P \uparrow}\,{\downarrow\uparrow}}{Q \supset P \uparrow}\,{\supset\uparrow^u}}{(P \supset Q) \vee (Q \supset P) \uparrow}\,{\vee\uparrow 2} \quad \dfrac{\dfrac{\dfrac{\dfrac{[\neg P \downarrow]_z \quad \dfrac{[P \downarrow]_v}{P \uparrow}\,{\downarrow\uparrow}}{F \downarrow}\,{\supset\downarrow}}{Q \uparrow}\,{F\downarrow}}{P \supset Q \uparrow}\,{\supset\uparrow^v}}{(P \supset Q) \vee (Q \supset P) \uparrow}\,{\vee\uparrow 1}}{(P \supset Q) \vee (Q \supset P) \uparrow}\,{\vee\downarrow^{y,z}}}{(P \vee \neg P) \supset ((P \supset Q) \vee (Q \supset P)) \uparrow}\,{\supset\uparrow^x}$$

Problem 2: Give a derivation, in sequent calculus, of the judgement:

$$\Longrightarrow (P \vee \neg P) \supset ((P \supset Q) \vee (Q \supset P))$$

Again, assume P and Q are atomic.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \vee \neg P), P, Q \Longrightarrow P}\,\mathit{init}}{(P \vee \neg P), P \Longrightarrow Q \supset P}\,{\supset}R}{(P \vee \neg P), P \Longrightarrow (P \supset Q) \vee (Q \supset P)}\,{\vee}R2 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{(P \vee \neg P), \neg P, P \Longrightarrow P}\,\mathit{init} \quad \dfrac{}{(P \vee \neg P), \neg P, P, F \Longrightarrow Q}\,FL}{(P \vee \neg P), \neg P, P \Longrightarrow Q}\,{\supset}L}{(P \vee \neg P), \neg P \Longrightarrow P \supset Q}\,{\supset}R}{(P \vee \neg P), \neg P \Longrightarrow (P \supset Q) \vee (Q \supset P)}\,{\vee}R1}{(P \vee \neg P) \Longrightarrow (P \supset Q) \vee (Q \supset P)}\,{\vee}L}{\Longrightarrow (P \vee \neg P) \supset ((P \supset Q) \vee (Q \supset P))}\,{\supset}R$$

2 Classical logic

In the section, you will examine the computational content of the sentence $(P \supset Q) \vee (Q \supset P)$.

Problem 1: Give a derivation, in classical logic, of the judgement:

$$(P \supset Q) \vee (Q \supset P)\ \mathsf{true}$$

If you wish, you may use the derived rules for elimination, which are reprised in the appendix. (This is not necessarily the easiest way, however.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P\ \mathsf{true}]_x}{Q \supset P\ \mathsf{true}}\supset T^y}{(P \supset Q) \vee (Q \supset P)\ \mathsf{true}}\vee T2 \quad [(P \supset Q) \vee (Q \supset P)\ \mathsf{false}]_k}{\#}\#}{Q\ \mathsf{true}}T\#^\ell}{P \supset Q\ \mathsf{true}}\supset T^x}{(P \supset Q) \vee (Q \supset P)\ \mathsf{true}}\vee T1 \quad [(P \supset Q) \vee (Q \supset P)\ \mathsf{false}]_k}{\#}\#}{(P \supset Q) \vee (Q \supset P)\ \mathsf{true}}T\#^k$$

Problem 2: Give the classical-logic proof term corresponding to your derivation. It is not required to rewrite your derivation using $M : A\ \mathsf{true}$ and $K : A\ \mathsf{false}$, but you may wish to do so.

$$k.(\mathsf{in}_1(\lambda x.\, \ell.(\mathsf{in}_2(\lambda y.\, x) \triangleright k)) \triangleright k)$$

Problem 3: Assume k_{init} is a continuation $(P \supset Q) \vee (Q \supset P)$ false. Use k_{init} to execute your proof term.

$$k.(\mathsf{in}_1(\lambda x.\, \ell.(\mathsf{in}_2(\lambda y.\, x) \triangleright k)) \triangleright k) \triangleright k_{init}$$
$$\rightarrow$$
$$\mathsf{in}_1(\lambda x.\, \ell.(\mathsf{in}_2(\lambda y.\, x) \triangleright k_{init})) \triangleright k_{init}$$

Problem 4: Explain in words the computational behavior of your proof term.

The term always says $P \supset Q$. If the resulting proof term for $P \supset Q$ is subsequently called with a proof term for P, it then time-travels and returns a second time, saying $Q \supset P$ instead.

3 Unprovability

Prove that *in constructive logic*, the judgement $(P \vee Q) \vee (P \supset Q)\ \mathsf{true}$ is not derivable. Assume that P and Q are atomic, and $P \neq Q$.

Suppose $(P \vee Q) \vee (P \supset Q)\mathsf{true}$ *is derivable (in natural deduction). By the completeness of sequent calculus, there exists a derivation of* $\Longrightarrow (P \vee Q) \vee (P \supset Q)$ *in sequent calculus. Let* $\mathcal{D}$ *be such a derivation.*

Only two rules apply for this goal, $\vee R1$ *and* $\vee R2$*. Thus* $\mathcal{D}$ *has the form:*

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Longrightarrow (P \vee Q)\end{array}}{\Longrightarrow (P \vee Q) \vee (P \supset Q)} \vee R1 \qquad \text{or} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2 \\ \Longrightarrow P \supset Q\end{array}}{\Longrightarrow (P \vee Q) \vee (P \supset Q)} \vee R2$$

Suppose the former. Then $\mathcal{D}_1$ *is a derivation of* $\Longrightarrow P \vee Q$*. Again, only two rules apply,* $\vee R1$ *and* $\vee R2$*. Thus* $\mathcal{D}_1$ *has the form:*

$$\dfrac{\begin{array}{c}\mathcal{D}_{1a} \\ \Longrightarrow P\end{array}}{\Longrightarrow P \vee Q} \vee R1 \qquad \text{or} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{1b} \\ \Longrightarrow Q\end{array}}{\Longrightarrow P \vee Q} \vee R2$$

But neither $\mathcal{D}_{1a}$ *nor* $\mathcal{D}_{1b}$ *can exist, since no rule applies.*

Thus, suppose the latter. Then $\mathcal{D}_2$ *is a derivation of* $\Longrightarrow P \supset Q$*. Only one rule applies for this goal, so* $\mathcal{D}_2$ *has the form:*

$$\dfrac{\begin{array}{c}\mathcal{D}_{2a} \\ P \Longrightarrow Q\end{array}}{\Longrightarrow P \supset Q} \supset R$$

But $\mathcal{D}_{2a}$ *cannot exist, since no rule applies. Therefore* $\mathcal{D}$ *cannot exist, and hence* $(P \vee Q) \vee (P \supset Q)\ \mathsf{true}$ *cannot be derivable.*

A Verifications & Uses

$$\frac{P\ \mathsf{atomic} \quad P\downarrow}{P\uparrow}\ \downarrow\uparrow \qquad \frac{A\uparrow \quad B\uparrow}{A\wedge B\uparrow}\ \wedge\uparrow \qquad \frac{A\wedge B\downarrow}{A\downarrow}\ \wedge\downarrow 1 \qquad \frac{A\wedge B\downarrow}{B\downarrow}\ \wedge\downarrow 2$$

$$\frac{\begin{array}{c}[A\downarrow]_x\\ \vdots\\ B\uparrow\end{array}}{A\supset B\uparrow}\ \supset\uparrow^x \qquad \frac{A\supset B\downarrow \quad A\uparrow}{B\downarrow}\ \supset\downarrow \qquad \frac{A\uparrow}{A\vee B\uparrow}\ \vee\uparrow 1 \qquad \frac{B\uparrow}{A\vee B\uparrow}\ \vee\uparrow 2$$

$$\frac{A\vee B\downarrow \quad \begin{array}{c}[A\downarrow]_x\\ \vdots\\ C\uparrow\end{array} \quad \begin{array}{c}[B\downarrow]_y\\ \vdots\\ C\uparrow\end{array}}{C\uparrow}\ \vee\downarrow^{x,y} \qquad \frac{}{T\uparrow}\ T\uparrow \qquad \frac{F\downarrow}{C\uparrow}\ F\downarrow$$

B Sequent Calculus rules

$$\frac{P\ \mathsf{atomic}}{\Delta, P\Longrightarrow P}\ \mathsf{init} \qquad \frac{\Delta\Longrightarrow A \quad \Delta\Longrightarrow B}{\Delta\Longrightarrow A\wedge B}\ \wedge R \qquad \frac{\Delta, A\Longrightarrow B}{\Delta\Longrightarrow A\supset B}\ \supset R$$

$$\frac{\Delta\Longrightarrow A}{\Delta\Longrightarrow A\vee B}\ \vee R1 \qquad \frac{\Delta\Longrightarrow B}{\Delta\Longrightarrow A\vee B}\ \vee R2 \qquad \frac{}{\Delta\Longrightarrow T}\ TR$$

$$\frac{\Delta, A\wedge B, A, B\Longrightarrow C}{\Delta, A\wedge B\Longrightarrow C}\ \wedge L \qquad \frac{\Delta, A\supset B\Longrightarrow A \quad \Delta, A\supset B, B\Longrightarrow C}{\Delta, A\supset B\Longrightarrow C}\ \supset L$$

$$\frac{\Delta, A\vee B, A\Longrightarrow C \quad \Delta, A\vee B, B\Longrightarrow C}{\Delta, A\vee B\Longrightarrow C}\ \vee L \qquad \frac{}{\Delta, F\Longrightarrow C}\ FL$$

C Classical Logic rules

$$\frac{\begin{array}{c}[A\ \mathsf{false}]_k\\ \vdots\\ \#\end{array}}{A\ \mathsf{true}}\ T\#^k \qquad \frac{\begin{array}{c}[A\ \mathsf{true}]_x\\ \vdots\\ \#\end{array}}{A\ \mathsf{false}}\ F\#^x \qquad \frac{A\ \mathsf{true} \quad A\ \mathsf{false}}{\#}$$

$$\frac{A\ \mathsf{true} \quad B\ \mathsf{true}}{A \land B\ \mathsf{true}}\ \land T \qquad \frac{A\ \mathsf{false}}{A \land B\ \mathsf{false}}\ \lor F1 \qquad \frac{B\ \mathsf{false}}{A \land B\ \mathsf{false}}\ \lor F2$$

$$\frac{\begin{array}{c}[A\ \mathsf{true}]_x \\ \vdots \\ B\ \mathsf{true}\end{array}}{A \supset B\ \mathsf{true}}\ \supset T^x \qquad \frac{A\ \mathsf{true} \quad B\ \mathsf{false}}{A \supset B\ \mathsf{false}}\ \supset F$$

$$\frac{A\ \mathsf{true}}{A \lor B\ \mathsf{true}}\ \lor T1 \qquad \frac{B\ \mathsf{true}}{A \lor B\ \mathsf{true}}\ \lor T2 \qquad \frac{A\ \mathsf{false} \quad B\ \mathsf{false}}{A \lor B\ \mathsf{false}}\ \lor F$$

$$\frac{}{T\ \mathsf{true}}\ TT \qquad \frac{}{F\ \mathsf{false}}\ FF \qquad \frac{A\ \mathsf{false}}{\neg A\ \mathsf{true}}\ \neg T \qquad \frac{A\ \mathsf{true}}{\neg A\ \mathsf{false}}\ \neg F$$

D Classical Logic proof terms

$$\frac{\begin{array}{c}[k : A\ \mathsf{false}] \\ \vdots \\ (M \triangleright K) : \#\end{array}}{k.(M \triangleright K) : A\ \mathsf{true}} \qquad \frac{\begin{array}{c}[x : A\ \mathsf{true}] \\ \vdots \\ (M \triangleright K) : \#\end{array}}{x.(M \triangleright K) : A\ \mathsf{false}} \qquad \frac{M : A\ \mathsf{true} \quad K : A\ \mathsf{false}}{(M \triangleright K) : \#}$$

$$\frac{M : A\ \mathsf{true} \quad N : B\ \mathsf{true}}{\langle M, N\rangle : A \land B\ \mathsf{true}} \qquad \frac{K : A\ \mathsf{false}}{\pi_1 \cdot K : A \land B\ \mathsf{false}} \qquad \frac{K : B\ \mathsf{false}}{\pi_2 \cdot K : A \land B\ \mathsf{false}}$$

$$\frac{\begin{array}{c}[x : A\ \mathsf{true}] \\ \vdots \\ M : B\ \mathsf{true}\end{array}}{\lambda x.M : A \supset B\ \mathsf{true}} \qquad \frac{M : A\ \mathsf{true} \quad K : B\ \mathsf{false}}{(M; K) : A \supset B\ \mathsf{false}}$$

$$\frac{M : A\ \mathsf{true}}{\mathsf{in}_1 M : A \lor B\ \mathsf{true}} \qquad \frac{M : B\ \mathsf{true}}{\mathsf{in}_2 M : A \lor B\ \mathsf{true}} \qquad \frac{K : A\ \mathsf{false} \quad L : B\ \mathsf{false}}{[K, L] : A \lor B\ \mathsf{false}}$$

$$\frac{}{() : T\ \mathsf{true}}\ TT \qquad \frac{}{\mathsf{abort} : F\ \mathsf{false}}\ FF \qquad \frac{K : A\ \mathsf{false}}{\neg K : \neg A\ \mathsf{true}} \qquad \frac{M : A\ \mathsf{true}}{\neg M : \neg A\ \mathsf{false}}$$

E Classical Logic reduction rules

$$
\begin{array}{lcl}
(k.(M' \triangleright K')) \triangleright K & \to & [K/k]M' \triangleright [K/k]K' \\
M \triangleright (x.(M' \triangleright K')) & \to & [M/x]M' \triangleright [M/x]K' \\
\langle M, N\rangle \triangleright \pi_1 \cdot K & \to & M \triangleright K \\
\langle M, N\rangle \triangleright \pi_2 \cdot K & \to & N \triangleright K \\
\lambda x.M \triangleright (N; K) & \to & [N/x]M \triangleright K \\
\mathsf{in}_1 M \triangleright [K, L] & \to & M \triangleright K \\
\mathsf{in}_2 M \triangleright [K, L] & \to & M \triangleright L \\
\neg K \triangleright \neg M & \to & M \triangleright K
\end{array}
$$

F Classical Logic derived elimination rules

$$
\frac{A \wedge B \ \mathsf{true}}{A \ \mathsf{true}} \wedge E1 \qquad \frac{A \wedge B \ \mathsf{true}}{B \ \mathsf{true}} \wedge E2 \qquad \frac{A \supset B \ \mathsf{true} \quad A \ \mathsf{true}}{B \ \mathsf{true}} \supset E
$$

$$
\frac{A \vee B \ \mathsf{true} \quad \begin{array}{c}[A \ \mathsf{true}]_x \\ \vdots \\ C \ \mathsf{true}\end{array} \quad \begin{array}{c}[B \ \mathsf{true}]_y \\ \vdots \\ C \ \mathsf{true}\end{array}}{C \ \mathsf{true}} \vee E^{x,y} \qquad \frac{F \ \mathsf{true}}{C \ \mathsf{true}} FE
$$

G Classical Logic proof terms for derived elimination rules

$$
\frac{M : A \wedge B \ \mathsf{true}}{k.(M \triangleright \pi_1 \cdot k) : A \ \mathsf{true}} \qquad \frac{M : A \wedge B \ \mathsf{true}}{k.(M \triangleright \pi_2 \cdot k) : B \ \mathsf{true}} \qquad \frac{M : A \supset B \ \mathsf{true} \quad N : A \ \mathsf{true}}{k.(M \triangleright (N; k)) : B \ \mathsf{true}}
$$

$$
\frac{M : A \vee B \ \mathsf{true} \quad \begin{array}{c}[x : A \ \mathsf{true}] \\ \vdots \\ N : C \ \mathsf{true}\end{array} \quad \begin{array}{c}[y : B \ \mathsf{true}] \\ \vdots \\ O : C \ \mathsf{true}\end{array}}{k.(M \triangleright [(x.(N \triangleright k)), (y.(O \triangleright k))]) : C \ \mathsf{true}} \qquad \frac{M : F \ \mathsf{true}}{k.(M \triangleright \mathsf{abort}) : C \ \mathsf{true}}
$$