

# Recitation 1: Simultaneous Induction

15-312: Principles of Programming Languages

Wednesday, January 18, 2017

## 1 Definitions

Below we will describe some simple judgements and take a look at proving properties for mutually defined judgements. This technique is called simultaneous induction.

*Some Judgements*

$$\frac{}{z \text{ even}} \text{ (EZ)} \quad \frac{n \text{ even}}{\text{succ}(n) \text{ odd}} \text{ (OS)} \quad \frac{n \text{ odd}}{\text{succ}(n) \text{ even}} \text{ (ES)}$$
$$\frac{}{\text{sum}(n, z, n)} \text{ (SZ)} \quad \frac{\text{sum}(a, b, c)}{\text{sum}(a, \text{succ}(b), \text{succ}(c))} \text{ (SS)}$$

## 2 Simultaneous Induction

Intuitively if recursions corresponds to induction, then mutual recursion corresponds to simultaneous induction. So, if we want to prove something about mutually defined judgements we need to prove two separate predicates.

Take the case of the judgements even and odd. Suppose we have two predicates, call them  $\mathcal{P}_{\text{even}}$  and  $\mathcal{P}_{\text{odd}}$ . To prove that  $\mathcal{P}_{\text{odd}}(n \text{ odd})$  and  $\mathcal{P}_{\text{even}}(n \text{ even})$  hold we need to prove the following:

- $\mathcal{P}_{\text{even}}(z \text{ even})$  holds.
- $\mathcal{P}_{\text{odd}}(\text{succ}(n) \text{ odd})$  holds assuming  $\mathcal{P}_{\text{even}}(n \text{ even})$  holds.
- $\mathcal{P}_{\text{even}}(\text{succ}(n) \text{ even})$  holds assuming  $\mathcal{P}_{\text{odd}}(n \text{ odd})$  holds.

## 3 Example

Let's prove the following theorem:

For all  $m, n, k$  if  $m$  even and  $\text{sum}(m, n, k)$ : if  $n$  even then  $k$  even, and if  $n$  odd then  $k$  odd. Let's also assume these lemmas.

**Lemma 1.** For all  $m, n, k, p$  if  $\text{sum}(m, n, k)$  and  $\text{sum}(m, n, p)$  then  $k = p$ .

**Lemma 2.** For all  $m, n, k$  if  $\text{sum}(m, \text{succ}(n), k)$  then  $k = \text{succ}(l)$  for some  $l$  such that  $\text{sum}(m, n, l)$ .

First we need to define our predicates, which we do as follows:

- $\mathcal{P}_{\text{even}}(n) = \text{“For all } m, k \text{ if } m \text{ even and } \text{sum}(m, n, k) \text{ then } k \text{ even”}$
- $\mathcal{P}_{\text{odd}}(n) = \text{“For all } m, k \text{ if } m \text{ even and } \text{sum}(m, n, k) \text{ then } k \text{ odd”}$

*Proof.* With this we go by rule induction on the judgements even and odd.

- Case:

$$\frac{}{z \text{ even}} \text{ (EZ)}$$

To Show: For all  $m, k$  if  $m$  even and  $\text{sum}(m, n, k)$  then  $k$  even

Let  $m, k$  be fixed and arbitrary.

To Show: if  $m$  even and  $\text{sum}(m, n, k)$  then  $k$  even 1) Assume  $m$  even

2) Assume  $\text{sum}(m, n, k)$

To Show:  $k$  even

3)  $\text{sum}(m, z, m)$

4)  $k = m$

To Show:  $m$  even

5)  $m$  even

By (SZ)

By Lemma 1 on 2) and 3)

By 1)

- Case:

$$\frac{n \text{ odd}}{\text{succ}(n) \text{ even}} \text{ (ES)}$$

To Show: For all  $m, k$  if  $m$  even and  $\text{sum}(m, \text{succ}(n), k)$  then  $k$  even

Let  $m, k$  be fixed and arbitrary.

To Show: if  $m$  even and  $\text{sum}(m, n, k)$  then  $k$  even

1) Assume  $m$  even

2) Assume  $\text{sum}(m, \text{succ}(n), k)$

To Show:  $k$  even

3)  $k = \text{succ}(p)$

4)  $\mathcal{P}_{\text{odd}}(n \text{ odd})$

5) For all  $m, k$  if  $m$  even and  $\text{sum}(m, n, k)$  then  $k$  odd

6)  $\text{sum}(m, n, p)$

7)  $p$  odd

8)  $\text{succ}(p)$  even

By Lemma 2 on 2)

By Induction Principle

By 4)

By Lemma 2 on 2)

By 5) on 1) and 6)

By (ES) on 7)

- Case:

$$\frac{n \text{ even}}{\text{succ}(n) \text{ odd}} \text{ (OS)}$$

Let as an exercise for the reader.

□