1. **Not Pirates and Not Gold**

(a) In lecture we developed a solution to this question:

How many nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 40$$

Solution: This is just pirates and gold with $n = 40, k = 5$. Thus, the answer is

$$\binom{40+5-1}{5-1} = \binom{44}{4}$$

(b) How many nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 40$$

Solution:
The variables $x_1 \ldots x_5$ will add to n with a leftover of $40 - n$. We can simply add another variable for whatever is leftover, and then we are simply counting pirates and gold with one more pirate. This gives a result of

$$\binom{40+6-1}{6-1} = \binom{45}{5}$$

(c) How many nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 40$$

if we must satisfy $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 \geq 5$.

Solution:
We can simply give the required value to each pirate before counting. We don't have any choice for them, so there is exactly one way to do this. Then, we distribute the remaining $40 - 5 - 4 - 3 - 2 - 1 = 25$ bars among the pirates. This give

$$\binom{25+5-1}{5-1} = \binom{29}{4}$$

(d) How many nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 40$$

if we must satisfy $x_1 \leq 20$

Solution:
We must have $x_1 \leq 20$ or $x_1 \geq 21$. We have just shown how to calculate this second value (with $\geq$ requirements), so we count by complement. Our result is

$$\binom{44}{4} - \binom{20+5-1}{5-1} = \binom{44}{4} - \binom{24}{4}$$

2. **There must be one...**

Let $S \subset \{0, 1, 2, 3, \ldots 99\}$ and $|S| = 10$.

Show that there must be two distinct subsets $A, B \subset S$ such that

$$\sum_{x \in A} x = \sum_{y \in B} y$$

.

Solution:
Every value in S is between 0 and 100, so the sum of any subset will be at least 0 and at most 1000 (we can show better bounds, but these will work).

Thus, there are at most 1000 different sums for any subset.

There are $2^{10} = 1024$ distinct subsets of S.

By pigeonhole principle, there must be two subsets with the same sum.

3. **Manhattaning Walks**

Consider the grid of points from $(0,0)$ to (n,n). Let $(a,x),(b,y),(c,z)$ be three points such that $0 < a < b < c < n$ and $0 < x < y < z < n$.

How many Manhattan walks are there from $(0,0)$ to (n,n) that don't go through any of the points $(a,x),(b,y),(c,z)$?

Solution:

We note that the number of paths to (n,n) which go through the point (p,q) is $\binom{p+q}{q}\binom{(n-p)+(n-q)}{n-q}$.

We then count using inclusion-exclusion. Let N denote the set of all paths to (n,n), and A,B,C be the paths which go through $(a,x),(b,y),(c,z)$ respectively.

Then the number of valid paths is

$$|N| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

where

$|N| = \binom{2n}{n}$

$|A| = \binom{a+x}{a}\binom{n-a+n-x}{n-a}$
$|B|, |C|$ follow similarly.

For $A \cap B$, we must first go to the point (a,x), then go from (a,x) to (b,y), then go from (b,y) to (n,n). Using the same reasoning, we see that

$|A \cap B| = \binom{a+x}{a}\binom{(b-a)+(y-x)}{b-a}\binom{(n-b)+(n-y)}{n-b}$
$|A \cap C|, |B \cap C|$ follow similarly.

Finally, $A \cap B \cap C$ requires us to go from each point to the next. Repeating our argument from above, we have
$|A \cap B \cap C| = \binom{a+x}{a}\binom{(b-a)+(y-x)}{b-a}\binom{(c-b)+(z-y)}{c-b}\binom{(n-c)+(n-z)}{n-c}$