15-251: Great Theoretical Ideas in Computer Science
Lecture 15
October 16, 2014

Algebra II: Fields, Polynomials

Recap: Definition of a group

G is a "**group** under operation •" if:

0. [Closure] G is closed under •
   i.e., a • b ∈ G ∀ a,b∈G

1. [Associativity] Operation • is associative:
   i.e., a • (b • c) = (a • b) • c ∀ a,b,c∈G

2. [Identity] There exists an element e∈G
   (called the "identity element") such that
   a • e = a, e • a = a ∀ a∈G

3. [Inverse] For each a∈G there is an element a^{-1}∈G
   (called the "inverse of a") such that
   $a \bullet a^{-1} = e$, $a^{-1} \bullet a = e$

Examples of groups

Any group of transformations is a group.

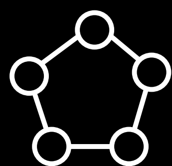
E.g., the 'mattress group' (AKA Klein 4-group)

•	Id	R	F	H
Id	Id	R	F	H
R	R	Id	H	F
F	F	H	Id	R
H	H	F	R	Id

identity element is Id

$R^{-1} = R$
$F^{-1} = F$
$H^{-1} = H$

Symmetries of undirected cycle: dihedral group



•	Id	r_1	r_2	r_3	r_4	f_1	f_2	f_3	f_4	f_5
Id	Id	r_1	r_2	r_3	r_4	f_1	f_2	f_3	f_4	f_5
r_1	r_1	r_2	r_3	r_4	Id	f_4	f_5	f_1	f_2	f_3
r_2	r_2	r_3	r_4	Id	r_1	f_2	f_3	f_4	f_5	f_1
r_3	r_3	r_4	Id	r_1	r_2	f_5	f_1	f_2	f_3	f_4
r_4	r_4	Id	r_1	r_2	r_3	f_3	f_4	f_5	f_1	f_2
f_1	f_1	f_3	f_5	f_2	f_4	Id	r_3	r_1	r_4	r_2
f_2	f_2	f_4	f_1	f_3	f_5	r_2	Id	r_3	r_1	r_4
f_3	f_3	f_5	f_2	f_4	f_1	r_4	r_2	Id	r_3	r_1
f_4	f_4	f_1	f_3	f_5	f_2	r_1	r_4	r_2	Id	r_3
f_5	f_5	f_2	f_4	f_1	f_3	r_3	r_1	r_4	r_2	Id

G =
{ Id, r_1, r_2, r_3, r_4,
f_1, f_2, f_3, f_4, f_5 }

Symmetries of directed cycle: Cyclic group

C_4

•	Id	R_1	R_2	R_3
Id	Id	R_1	R_2	R_3
R_1	R_1	R_2	R_3	Id
R_2	R_2	R_3	Id	R_1
R_3	R_3	Id	R_1	R_2

In a group table, every row and every column is a permutation of the group elements
Follows because (i) a b = a c ⇒ b = c
(ii) b a = c a ⇒ b = c

Abelian groups

In a group we do NOT NECESSARILY have

a • b = b • a

Definition:
"a,b ∈ G commute" means ab = ba.

Definition:
A group is said to be abelian if all pairs a,b ∈ G commute.

Order of a group element

Let G be a ***finite*** group. Let $a \in G$.

Definition: The order of x, denoted ord(a), is the smallest $m \geq 1$ such that $a^m = 1$.

Note that $a, a^2, a^3, \ldots, a^{m-1}, a^m=1$ all distinct.

Order Theorem: For every $a \in G$, ord(a) divides |G|.

Corollary: $a^{|G|}=1$ for all $a \in G$.

Corollary (Euler's Theorem): For $a \in Z_n^*$, $a^{\phi(n)} = 1$
That is, if gcd(a,n)=1, then $a^{\phi(n)} \equiv 1 \pmod{n}$

Corollary (Fermat's little theorem):
For prime p, if gcd(a,p)=1, then
$a^{p-1} \equiv 1 \pmod{p}$

Cyclic groups

A finite group G of order n is cyclic if
$G= \{e,b,b^2,\ldots,b^{n-1}\}$ for some group element b

In such a case, we say b ***"generates"*** G,
or b is a *"generator"* of G.

Examples:

- $(Z_n, +)$ (1 is a generator)
- C_4 (Rot_{90} is a generator)

Non-examples: Mattress group;
any non-abelian group.

How many generators does $(Z_n, +)$ have?

Answer: $\Phi(n)$

b generates $Z_n \Leftrightarrow \exists$ a s.t. $ba \equiv 1 \pmod{n}$
($ba = b+b+\ldots+b$ (a times))

Same holds for *any* cyclic group with n elements

Subgroups

Q: Is (Even integers, +) a group?

A: Yes. It is a "subgroup" of $(\mathbb{Z},+)$

Definition: Suppose $(G, \bullet)$ is a group.
If $H \subseteq G$, and if $(H,\bullet)$ is also a group,
then H is called a subgroup of G.

To check H is a subgroup of G, check:

1. H is closed under $\bullet$
2. $e \in H$
3. If $h \in H$ then $h^{-1} \in H$
   - (3rd condition follows from 1,2 if H is finite)

Examples

Every G has two trivial subgroups: {e}, G
Rest are called "proper" subgroups

Suppose k, $1 < k < n$, divides n.

Q1. Is $(\{0, k, 2k, 3k, \ldots, (n/k-1)k\}, +_n)$ subgroup of $(Z_n,+_n)$?
Yes!

Q2. Is $(Z_k, +_k)$ a subgroup of $(Z_n, +_n)$?
No! it doesn't even have the same operation

Q3. Is $(Z_k, +_n)$ a subgroup of $(Z_n, +_n)$?
No! Z_k is not closed under $+_n$

Lagrange's Theorem

Theorem: If G is a finite group, and H is a subgroup then |H| divides |G|.

Proof similar to order theorem.

Corollary (order theorem): If $x \in G$, then ord(x) divides |G|.

Proof of Corollary:

Consider the set $T_x = (x, x^2, x^3, \ldots)$

(i) ord(x) = $|T_x|$

(ii) $(T_x, \bullet)$ is a subgroup of $(G, \bullet)$ (check!)

A useful corollary

If G is a finite group and
H is a proper subgroup of G, then $|H| \leq |G|/2$

Example:
$G = (\{0,1\}^n, +)$ (+ : coordinate-wise addition mod 2)

$H = \{(x_1,x_2,\ldots,x_n) : x_1+x_2+ \ldots + x_n = 0\}$

H is a proper subgroup (check!)
So $|H| \leq 2^{n-1}$
(in fact, in this case, $|H| = 2^{n-1}$)

It was nice meeting you, groups!

Number Theory Interlude II

Bezout's identity

Let a,b be arbitrary positive integers.
There exist integers r and s such that

$r\,a + s\,b = \gcd(a,b)$

Follows from Extended Euclid Algorithm

A non-algorithmic proof:

- Consider the set L of all *positive* integers that can be expressed as r a + s b for some integers r,s.
- L is non-empty (eg. $a \in S$)
- So L has a minimum element d

(well-ordering principle ⇔ principle of induction)

Claim: $d = \gcd(a,b)$

Claim: gcd(a,b) = d (the minimum positive integer expressible as ra+sb)

1. gcd(a,b) divides both a and b, and hence also divides d. So $d \geq \gcd(a,b)$

2. d divides both a and b, and hence $d \leq \gcd(a,b)$

Let's show d | a.
Write a = q d + t , with $0 \leq t < d$.

t = a – q d is also expressible as a combination r' a + s' b.
Contradicts minimality of d.

Extended Euclid & Unique Factorization

Lemma: If gcd(a,b)=1 and a | bc, then a | c.

Proof: Let r,s be such that r a + s b =1

r a c + s b c = c

a | bc and a | r a c, so a | c. □

Corollary: If p is a prime and $p \mid q_1 q_2 \ldots q_k$, then p must divide some q_i.

If the q_i's are also prime, then $p = q_i$ for some i.

Unique prime factorization follows from this!

Extended Euclid and Chinese Remaindering

Chinese Remainder Theorem: Suppose $n_1, n_2, \ldots, n_k$ are pairwise coprime. Then, for all integers $a_1, a_2, \ldots, a_k$, there exists an integer x solving the below system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k}. \end{aligned}$$

Further, all solutions x are congruent modulo $N = \prod_{i=1}^{k} n_i$.

Uniqueness of solutions modulo N

If x,y are two solutions, then n_i divides x-y, for i=1,2,…k

Since the n_i are coprime, this means the product $N = n_1 n_2 \ldots n_k$ divides (x-y), thus $x \equiv y \pmod{N}$

Extended Euclid and Chinese Remaindering

Chinese Remainder Theorem: Suppose $n_1, n_2, \ldots, n_k$ are pairwise coprime. Then, for all integers $a_1, a_2, \ldots, a_k$, there exists an integer x solving the below system of simultaneous congruences

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Further, all solutions x are congruent modulo $N = \prod_{i=1}^{k} n_i$.

Proof for k=2:

Take $x = a_1 (n_2^{-1} \bmod n_1)\, n_2 + a_2 (n_1^{-1} \bmod n_2)\, n_1$

Can compute x efficiently (by computing modular inverses)

Extended Euclid and Chinese Remaindering

Chinese Remainder Theorem: Suppose $n_1, n_2, \ldots, n_k$ are pairwise coprime. Then, for all integers $a_1, a_2, \ldots, a_k$, there exists an integer x solving the below system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k}. \end{aligned}$$

Further, all solutions x are congruent modulo $N = \prod_{i=1}^{k} n_i$.

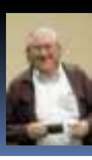
For arbitrary k: Let $m_i = N/n_i$ Note $\gcd(m_i, n_i) = 1$, $n_i \mid m_j$ for $j \neq i$

Take $x = a_1 (m_1^{-1} \bmod n_1)\, m_1 + a_2 (m_2^{-1} \bmod n_2)\, m_2 + \ldots + a_k (m_k^{-1} \bmod n_k)\, m_k$

Field Theory

Find out about the wonderful world of $\mathbb{F}_{2^k}$
where two equals zero, plus is minus,
and squaring is a linear operator!

– Richard Schroeppel



A group is a set with a single binary operation.

Number-theoretic sets often have more than one operation defined on them.

For example, in $\mathbb{Z}$, we can do both addition and multiplication.

Same in Z_n (we can add and multiply modulo n)

For reals $\mathbb{R}$ or rationals $\mathbb{Q}$, we can also divide (inverse operation for multiplication).

Fields

Informally, it's a place where you can add, subtract, multiply, and divide.

Examples:

Real numbers	$\mathbb{R}$	
Rational numbers	$\mathbb{Q}$	
Complex numbers	$\mathbb{C}$	
Integers mod ***prime***	Z_p	(Why?)

NON-examples:

Integers $\mathbb{Z}$		division??
Non-negative reals $\mathbb{R}^+$		subtraction??

Field – formal definition

A ***field*** is a set F with *two* binary operations, called + and •.

(F,+) an abelian group, with identity element called 0

(F \ {0},•) an abelian group, identity element called 1

Distributive Law holds:
a•(b+c) = a•b + a•c

Example: $\mathbb{F}_3 = Z_3^*$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Example

Quadratic "number field"

$\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} : a,b \in \mathbb{Q} \}$

Addition: $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$

Multiplication:
$(a + b\sqrt{2}) \bullet (c + d\sqrt{2}) = (ac+2bd) + (ad+bc)\sqrt{2}$

Exercise: Prove above defines a field.

Finite fields

Some familiar *infinite* fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (now $\mathbb{Q}(\sqrt{2})$)

Finite fields we know: Z_p aka $\mathbb{F}_p$ for p a prime

Is there a field with 2 elements? Yes
Is there a field with 3 elements? Yes
Is there a field with 4 elements? Yes

$\mathbb{F}_4$

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Evariste Galois (1811−1832) introduced the concept of a finite field (also known as a Galois Field in his honor)

Finite fields

Is there a field with 2 elements? Yes
Is there a field with 3 elements? Yes
Is there a field with 4 elements? Yes
Is there a field with 5 elements? Yes
Is there a field with 6 elements? No
Is there a field with 7 elements? Yes
Is there a field with 8 elements? Yes
Is there a field with 9 elements? Yes
Is there a field with 10 elements? No

Finite fields

Theorem (which we won't prove):

There is a field with q elements
if and only if q is a power of a prime.

Up to *isomorphism*, it is unique.

That is, all fields with q elements have the same addition and multiplication tables, after renaming elements.

This field is denoted $\mathbb{F}_q$ (also GF(q))

Finite fields

Question:

If q is a prime power but not just a prime, what ***are*** the addition and multiplication tables of $\mathbb{F}_q$?

Answer:

It's a bit hard to describe.

We'll tell you later, but for 251's purposes, you only need to know about prime q.

Polynomials

Polynomials

Informally, a polynomial is an expression that looks like this:

$$6x^3 - 2.3x^2 + 5x + 4.1$$

x is a symbol, called the *variable*

the 'numbers' standing next to powers of x are called the *coefficients*

Polynomials

Informally, a polynomial is an expression that looks like this:

$$6x^3 - 2.3x^2 + 5x + 4.1 \in \mathbb{R}[x]$$

Actually, coefficients can come from any *field*.

Can allow multiple variables, but we won't.

Set of polynomials with variable x and coefficients from field F is denoted **F[x]**.

Polynomials – formal definition

Let F be a field and let x be a variable symbol.

F[x] is the set of polynomials over F,

defined to be expressions of the form

$$c_d x^d + c_{d-1} x^{d-1} + \cdots + c_2 x^2 + c_1 x + c_0$$

where each c_i is in F, and $c_d \neq 0$.

We call d the degree of the polynomial.

Also, the expression 0 is a polynomial.

(By convention, we call its degree $-\infty$.)

Adding and multiplying polynomials

You can add and multiply polynomials.

Example. Here are two polynomials in $\mathbb{F}_{11}[x]$

$P(x) = x^2 + 5x - 1$
$Q(x) = 3x^3 + 10x$

$$P(x) + Q(x) = 3x^3 + x^2 + 15x - 1$$
$$= 3x^3 + x^2 + 4x - 1$$
$$= 3x^3 + x^2 + 4x + 10$$

Adding and multiplying polynomials

You can add and multiply polynomials (they are a "ring" but we'll skip a formal treatment of rings)

Example. Here are two polynomials in $\mathbb{F}_{11}[x]$

$P(x) = x^2 + 5x - 1$
$Q(x) = 3x^3 + 10x$

$$P(x) \cdot Q(x) = (x^2 + 5x - 1)(3x^3 + 10x)$$
$$= 3x^5 + 15x^4 + 7x^3 + 50x^2 - 10x$$
$$= 3x^5 + 4x^4 + 7x^3 + 6x^2 + x$$

Adding and multiplying polynomials

Polynomial addition is associative and commutative.
$0 + P(x) = P(x) + 0 = P(x)$.
$P(x) + (-P(x)) = 0$.
So (F[x], +) is an abelian group!

Polynomial multiplication is associative and commutative.
$1 \cdot P(x) = P(x) \cdot 1 = P(x)$.
Multiplication distributes over addition:
$P(x) \cdot (Q(x) + R(x)) = P(x) \cdot Q(x) + P(x) \cdot R(x)$

If P(x) / Q(x) were always a polynomial, then F[x] would be a field! Alas…

Dividing polynomials?

P(x) / Q(x) is not necessarily a polynomial.

So F[x] is not quite a field.
(It's an "integral domain")

Same with $\mathbb{Z}$, the integers:
it has everything except division.

Actually, there are many analogies between F[x] and $\mathbb{Z}$.
- starting point for rich interplay between algebra, arithmetic, and geometry in modern mathematics

Dividing polynomials?

$\mathbb{Z}$ has the concept of "division with remainder":

Given $a,b \in \mathbb{Z}$, $b \neq 0$, can write
$a = q \cdot b + r$,
where r is "smaller than" b.

F[x] has the same concept:

Given $A(x),B(x) \in F[x]$, $B(x) \neq 0$, can write
$A(x) = Q(x) \cdot B(x) + R(x)$,
where $\deg(R(x)) < \deg(B(x))$.

"Division with remainder" for polynomials

Example: Divide $6x^4+8x+1$ by $2x^2+4$ in $\mathbb{F}_{11}[x]$

```
           3x²+5
       ____________
2x²+4 | 6x⁴+8x+1
      - 6x⁴+x²
        ________
             −x²+8x+1
           − −x²+9
             ________
               8x+3
```

Check:
$6x^4+8x+1$
$= (3x^2+5)(2x^2+4)+(8x+3)$
(in $\mathbb{F}_{11}[x]$)

Integers $\mathbb{Z}$

"size" = abs. value

"division":
$a = qb+r, \quad |r| < |b|$

can use Euclid's Algorithm
to find GCDs

p is "prime":
no nontrivial divisors

$\mathbb{Z}$ mod p:
a field iff p is prime

Polynomials F[x]

"size" = degree

"division":
$A(x) = Q(x)B(x)+R(x)$,
$\deg(R) < \deg(B)$

can use Euclid's Algorithm
to find GCDs

P(x) is "irreducible":
no nontrivial divisors

F[x] mod P(x):
a field iff P(x) is irreducible
(with $|F|^{\deg(P)}$ elements)

The field with 4 elements

Degree < 2 polynomials $\{0,1,x,1+x\} \subseteq \mathbb{F}_2[x]$

Addition and multiplication modulo $1+x+x^2$

$\mathbb{F}_4$

a=x
b=1+x

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Enough algebraic theory.

Let's play with polynomials!

Evaluating polynomials

Given a polynomial $P(x) \in F[x]$,
P(a) means its evaluation at element a.

E.g., if $P(x) = x^2+3x+5$ in $\mathbb{F}_{11}[x]$

$P(6) = 6^2+3\cdot6+5 = 36+18+5 = 59 = 4$

$P(4) = 4^2+3\cdot4+5 = 16+12+5 = 33 = 0$

Definition: α is a **root** of P(x) if $P(\alpha) = 0$.

Polynomial roots

Theorem:

Let $P(x) \in F[x]$ have degree 1.
Then P(x) has exactly 1 root.

Proof:

Write $P(x) = cx + d$ (where $c \neq 0$).
Then $P(r) = 0 \iff cr + d = 0$
$\iff cr = -d$
$\iff r = -d/c$.

Polynomial roots

Theorem:

Let $P(x) \in F[x]$ have degree 2.
Then P(x) has… how many roots??

E.g.: x^2+1…

\# of roots over $\mathbb{F}_2[x]$: 1 (namely, 1)

\# of roots over $\mathbb{F}_3[x]$: 0

\# of roots over $\mathbb{F}_5[x]$: 2 (namely, 2 and 3)

\# of roots over $\mathbb{R}[x]$: 0

\# of roots over $\mathbb{C}[x]$: 2 (namely, i and −i)

The single most important theorem about polynomials over fields:

A nonzero degree-d polynomial has at most d roots.

Theorem: Over a field, for all d ≥ 0, a nonzero degree-d polynomial P has at most d roots.

Proof by induction on $d \in \mathbb{N}$:

Recall our convention: $\deg(0) = -\infty$

Base case: If P(x) is degree-0 then P(x) = a for some a≠0. This has 0 roots.

Induction:

Assume true for d ≥ 0. Let P(x) have degree d+1.

If P(x) has 0 roots: we're done! Else let b be a root.

Divide with remainder: $P(x) = Q(x)(x-b) + R(x)$. (∗)

$\deg(R) < \deg(x-b) = 1$, so R(x) is a constant. Say $R(x)=r$.

Plug x = b into (∗): $0 = P(b) = Q(b)(b-b)+r = 0+r = r$.

So $P(x) = Q(x)(x-b)$. Now, $\deg(Q) = d$. ∴ Q has ≤ d roots.

∴ P(x) has ≤ d+1 roots, completing the induction.

A useful corollary

Theorem: Over a field F, for all d ≥ 0, degree-d polynomials have at most d roots.

Corollary: Suppose a polynomial $R(x) \in F[x]$ is such that

(i) R has degree ≤ d and
(ii) R has > d roots

Then R must be the 0 polynomial

I've used the above corollary *several times* in my research.

Theorem:

Over a field, degree-d polynomials have at most d roots.

Reminder:

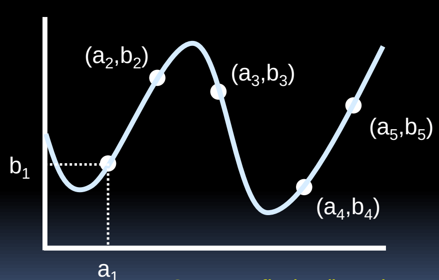
This is only true over a field.

E.g., consider $P(x) = 3x$ over Z_6.

It has degree 1, but 3 roots: 0, 2, and 4.

Interpolation

Say you're given a bunch of "data points"



Can you find a (low-degree) polynomial which "fits the data"?

Interpolation

Let pairs $(a_1,b_1), (a_2,b_2), \ldots, (a_{d+1},b_{d+1})$ from a field F be given (with all a_i's distinct).

Theorem:

There is exactly one polynomial P(x) of degree at most d such that $P(a_i) = b_i$ for all i = 1…d+1.

E.g., through 2 points there is a unique linear polynomial.

Interpolation

There are two things to prove.

1. There is at *least* one polynomial of degree $\leq$ d passing through all d+1 data points.

2. There is at *most* one polynomial of degree $\leq$ d passing through all d+1 data points.

Let's prove #2 first.

Interpolation

Theorem: Let pairs $(a_1,b_1), (a_2,b_2), \ldots, (a_{d+1},b_{d+1})$ from a field F be given (with all a_i's distinct).
Then there is at most one polynomial P(x) of degree at most d with $P(a_i) = b_i$ for all i.

Proof: Suppose P(x) and Q(x) both do the job.
Let R(x) = P(x)−Q(x).
Since deg(P), deg(Q) $\leq$ d we must have deg(R) $\leq$ d.
But $R(a_i) = b_i - b_i = 0$ for all i = 1…d+1.
Thus R(x) has more roots than its degree.
∴ R(x) must be the 0 polynomial, i.e., P(x)=Q(x).

Interpolation

Now let's prove the other part,
that there is at least one polynomial.

Theorem:
Let pairs $(a_1,b_1), (a_2,b_2), \ldots, (a_{d+1},b_{d+1})$ from a field F be given (with all a_i's distinct).
Then there exists a polynomial P(x) of degree at most d with $P(a_i) = b_i$ for all i.

Interpolation

The method for constructing the polynomial
is called Lagrange Interpolation.

Discovered in 1779
by Edward Waring.

Rediscovered in 1795
by J.-L. Lagrange.

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

Want P(x)
(with degree $\leq$ d)
such that $P(a_i) = b_i$ $\forall$i.

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Can we do this special case?

Promise: once we solve this special case,
the general case is very easy.

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Just divide P(x) by this number.

Idea #1: $P(x) = (x-a_2)(x-a_3)\cdots(x-a_{d+1})$

Degree is d. ✔

$P(a_2) = P(a_3) = \cdots = P(a_{d+1}) = 0$. ✔

$P(a_1) = (a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1})$. ??

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Numerator is a deg. d polynomial

Denominator is a nonzero field element

Idea #2:

$$S_1(x) = \frac{(x-a_2)(x-a_3)\cdots(x-a_{d+1})}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1})}$$

Call this the selector polynomial for a_1.

Lagrange Interpolation

a_1	0
a_2	1
a_3	0
…	…
a_d	0
a_{d+1}	0

Great! But what about this data?

$$S_2(x) = \frac{(x-a_1)(x-a_3)\cdots(x-a_{d+1})}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{d+1})}$$

Lagrange Interpolation

a_1	0
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	1

Great! But what about this data?

$$S_{d+1}(x) = \frac{(x-a_1)(x-a_2)\cdots(x-a_d)}{(a_{d+1}-a_1)(a_{d+1}-a_2)\cdots(a_{d+1}-a_d)}$$

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

Great! But what about this data?

$$P(x) = b_1 \cdot S_1(x) + b_2 \cdot S_2(x) + \cdots + b_{d+1} \cdot S_{d+1}(x)$$

Lagrange Interpolation – example

Over Z_{11}, find a polynomial P of degree ≤ 2 such that P(5) = 1, P(6) = 2, P(7) = 9.

$S_5(x) = 6\,(x-6)(x-7)$ $\qquad \frac{1}{(5-6)(5-7)}$

$S_6(x) = -1\,(x-5)(x-7)$

$S_7(x) = 6\,(x-5)(x-6)$

$P(x) = 1\,S_5(x) + 2\,S_6(x) + 9\,S_7(x)$

$P(x) = 6(x^2-13x+42) - 2(x^2-12x+35) + 54(x^2-11x+30)$

$P(x) = 3x^2+x+9$

The Chinese Remainder Theorem had a very similar proof

Not a coincidence:

algebraically, integers & polynomials share many common properties

Lagrange interpolation is the *exact analog* of Chinese Remainder Theorem for polynomials.

Chinese Remainder Theorem: Suppose $n_1, n_2, \ldots, n_k$ are pairwise coprime. Then, for all integers $a_1, a_2, \ldots, a_k$, there exists an integer x solving the below system of simultaneous congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k}. \end{aligned}$$

Further, all solutions x are congruent modulo $N = \prod_{i=1}^{k} n_i$.

Let $m_i = N/n_i$

i'th "selector" number: $T_i = (m_i^{-1} \bmod n_i)\, m_i$

$$x = a_1 T_1 + a_2 T_2 + \ldots + a_k T_k$$

Recall: Interpolation

Let pairs $(a_1,b_1), (a_2,b_2), \ldots, (a_{d+1},b_{d+1})$ from a field F be given (with all a_i's distinct).

Theorem:

There is a unique degree d polynomial P(x) satisfying $P(a_i) = b_i$ for all i = 1…d+1.

A linear algebra view

Let $p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_d x^d$

Need to find the coefficient vector $(p_0, p_1, \ldots, p_d)$

$p(a) = p_0 + p_1 a + \ldots + p_d a^d$

$\quad = 1 \cdot p_0 + a \cdot p_1 + a^2 \cdot p_2 + \ldots + a^d \cdot p_d$

Thus we need to solve:

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{d+1} \end{pmatrix}$$

Lagrange interpolation

The $(d+1) \times (d+1)$ Vandermonde matrix

$$M = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix}$$

is **invertible.**

- The determinant of M is nonzero when a_i's are distinct.

Thus can recover coefficient vector **as** $\vec{p} = M^{-1}\vec{b}$

The columns of M^{-1} are given by the coefficients of the various "selector" polynomials we constructed in Lagrange interpolation.

Representing Polynomials

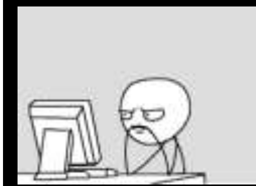
Let P(x)∈F[x] be a degree-d polynomial.

Representing P(x) using d+1 field elements:

1. List the d+1 coefficients.
2. Give P's value at d+1 different elements.

Rep 1 to Rep 2: Evaluate at d+1 elements

Rep 2 to Rep 1: Lagrange Interpolation



Study Guide

Group Theory:
- Abelian
- Order theorem
- Isomorphism
- Subgroups

Number Theory:
- Euler's theorem
- Chinese Remainder theorem

Fields:
- Definitions
- Examples
- Finite fields of prime order

Polynomials:
- Degree-d polys have ≤ d roots.
- Polynomial division with remainder
- Lagrange Interpolation