MATHEMATICAL GAMES

Plan

Introduction to Impartial Combinatorial Games

The Beer Can Game

Players alternate placing beer cans on a circular table. Once placed a can cannot move. The first one who cannot put a can on the table loses.

What's the winning strategy?

MIRRORING STRATEGY

First player wins

A Take-Away Game

Two Players: 1 and 2

A move consists of removing one, two, or three chips from the pile

Players alternate moves, with Player 1 starting

Player that removes the last chip wins

21 chips

Which player would you rather be?

Try Small Examples!

If there are 1, 2, or 3 only, player who moves next wins

If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips, and his opponent will win

With 5, 6 or 7 chips left, the player who moves next can win by leaving 4 chips

21 chips

0, 4, 8, 12, 16, … are target positions; if a player moves to that position, they can win the game

Therefore, with 21 chips, Player 1 can win!

What if the last player to move loses?

If there is 1 chip, the player who moves next loses

If there are 2, 3, or 4 chips left, the player who moves next can win by leaving only 1

In this case, 1, 5, 9, 13, … are a win for the second player

Combinatorial Games

• A set of positions (position = state of the game)
• Two players (know the state)
• Rules specify for each position which moves to other positions are legal moves
   • we restrict to "impartial" games (same moves available for both players)
• The players alternate moving
• A terminal position is one in which there are no moves
• The game must reach terminal position and end in a finite number of moves.
   • (No draws!)

Normal Versus Misère

Games ends by reaching a terminal position from which there are no moves.

Normal Play Rule: The last player to move wins

Misère Play Rule: The last player to move loses

What is not captured

No randomness
   (This rules out poker)
No hidden state
   (This rules out Battleship)
No draws
   (This rules out tic-tac-toe)

However, Go, Hex and many other games do fit.

P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved the game into that position)
   • Sometimes called LOSING positions

N-Position: Positions that are winning for the Next player (the player who is about to move from the current position)
   • Sometimes called WINNING positions

21 chips

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position ("First player wins")

What's a P-Position?

"Positions that are winning for the Previous player (the player who just moved)"

That means (focusing on normal play rule):

For any move that N makes

  There exists a move for P such that

    For any move that N makes

   There exists a move for P such that

        ⋮

      There exists a move for P such that

        There are no possible moves for N

P-positions and N-positions can be defined recursively by the following:

(1) All terminal positions (from which there are no moves) are P-positions (normal winning rule)

(2) A position where *at least 1* move leads to a P-position is an N-position.

(3) A position where *every possible move* leads to an N-position is a P-position

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Game tree visualization

Player A   Player B   Terminal Configuration
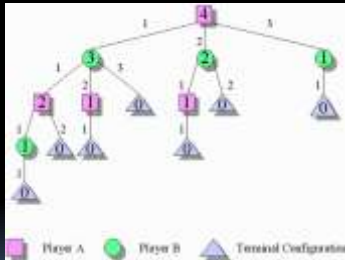
Can label nodes as P or N working upwards from leaves (which are P-positions)
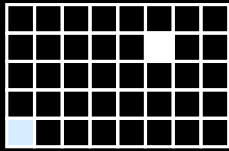

Game tree visualization

Theorem: Every position in a combinatorial game is either a P-position or an N position.

Proof: Follows from the recursive labeling algorithm, and the fact that the game must end.

(Formally, induction on maximum number of steps for game to end from given position; the "height" of the position in game tree.)

Chomp!

Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

BUT -- You cannot move to (1,1)

Show That This is a P-position
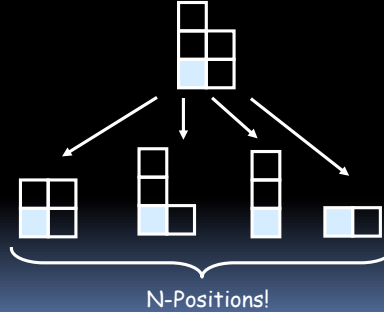
N-Positions!

Show that this is an N-position

P-position!

Let's Play! I'm player 1

No matter what you do, I can mirror it!

Mirroring is an extremely important strategy in combinatorial games!

Theorem: A square starting position of Chomp is an N-position (Player 1 can win)

Proof:

The winning strategy for player 1 is to chomp on (2,2), leaving only an "L" shaped position

Then, for any move that Player 2 takes, Player 1 can simply mirror it on the flip side of the "L"

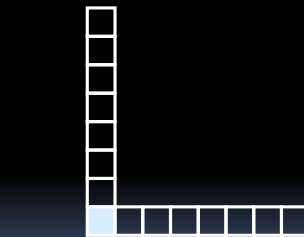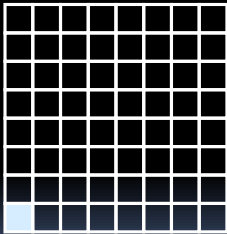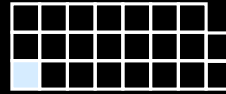**Theorem:** *Every* rectangle of area > 1 is an N-position

Proof: Consider this position:

This is either a P or an N-position.
If it's a P-position, then the original rectangle was N.
If it's an N-position, then there exists a move X from it to a P-position.

But by the geometry of the situation, X is also available as a move from the starting rectangle. It follows that the original rectangle is an N-position. **QED**

Note that this is a *non-constructive* proof.
We've shown that there exists a winning move from a rectangle, but we have not found the move!

The Game of Nim

Two players take turns moving

Winner is the last player to remove chips

A move consists of selecting a pile and removing one or more chips from it.

(In one move, you cannot remove chips from more than one pile.)

Analyzing Simple Positions

We use (x,y,z) to denote this position

(0,0,0) is a: P-position

One-Pile Nim

What happens in positions of the form (x,0,0)? (with x>0)

The first player can just take the entire pile, so (x,0,0) is an N-position

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Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips.

If it is the opponent's turn to move from such a position, he must change to a position in which the two piles have different number of chips.

From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position).   [Mirroring again!]

Two-Pile Nim

<u>Theorem</u>: P-positions are those for which the two piles have an equal number of chips.
(and N-positions ≡ unequal piles)

3-step process to prove this formally:

1. Check terminal positions have equal piles.
2. Show that from unequal piles, there is some move making the piles equal.
3. Show that all moves from equal piles result in a position with unequal piles.

Nim-Sum

The nim-sum of two non-negative integers is their addition *without carry* in base 2.

We will use $\oplus$ to denote the nim-sum

$$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$$
$$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$$
$$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer x,

$$x \oplus x = 0$$

Cancellation Property Holds

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$

3-Pile Nim

Bouton's Theorem: A position (x,y,z) in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:

Let Z denote the set of Nim positions with nim-sum zero

Let NZ denote the set of Nim positions with non-zero nim-sum

We prove the theorem by proving that Z and NZ satisfy the three conditions of P-positions and N-positions

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(1) All terminal positions are in Z

The only terminal position is (0,0,0)

(2) From each position in NZ, there is a move to a position in Z

```
001010001          001010001
100010111          100010111
⊕ 111010000    ⊕  101000110
010010110          000000000
```

Look at leftmost column with an odd # of 1s

Rig one of the numbers with a 1 in that column so that everything adds up to zero

(3) Every move from a position in Z is to a position in NZ

If (x,y,z) is in Z, and x is changed to $x' \neq x$, then we cannot have

$$x \oplus y \oplus z = 0 = x' \oplus y \oplus z$$

Because then $x = x'$ QED

Nim as a sum

3-pile nim is like the "sum" of three 1-pile nim games.

At each step, pick one of the three games (that is non-terminal) and make a move in that game.

The position is terminal iff all 3 games are in terminal positions.

Sum of games

A and B are games. The game A+B is a new game where the allowed moves are to pick one of the two games A or B (that is in a non-terminal position) and make a legal move in that game.

Terminal positions of A+B ≡ Positions that terminate in *both* A and B.

Note: The sum operator is
- commutative [A+B = B+A]
- associative [(A+B) + C = A + (B+C)]

Analyzing Games

We assign a number to positions of any (normal, impartial) game.

This number is called the Nimber of the game.*

- Also called the "Sprague-Grundy" number of a game.

*We will blur the distinction between game positions and games. For each position, we consider the game resulting when that position is the starting position.

The MEX

The "MEX" of a finite set of natural numbers is the Minimum EXcluded element.

MEX {0, 1, 2, 4, 5, 6} =   3

MEX {1, 3, 5, 7, 9} =   0

MEX {} =   0

Definition of Nimber

The Nimber of a game (position) G (denoted $N(G)$) is defined inductively as follows:

$N(G) = 0$ if G is terminal

$N(G) = MEX\{N(G_1), N(G_2), \dots N(G_n)\}$

where $G_1, G_2, \dots G_n$ are the successor positions of G (i.e. the positions resulting from all the allowed moves)

Nim revisited

Let P_k denote the position that is a pile of k stones in the game of (one-pile) Nim.

Theorem: $N(P_k) = k$

Theorem: $N(P_k) = k$

Proof: Use induction.

Base case is when k=0. Trivial.

When k>0 the set of possible positions after a legal move is P_{k-1}, P_{k-2}, ... P_0.

By induction these positions have nimbers k-1, k-2, ... 0.

The MEX of these is k.   QED.

Theorem: A game/position G is a P-position *if and only if* $N(G)=0$.

(i.e., Nimber = 0 if & only if P-position)

Proof: By induction (on max. # steps for game to end from given position)

Trivially true if G is a terminal position.

Suppose G is non-terminal.

(only if) To prove: $N(G) \neq 0$ implies N-position

If $N(G) \neq 0$, then by the MEX rule there exists a move to G' from G that has $N(G')=0$.

By induction this is a P-position.
Thus G is an N-position.

Nimber = 0 iff P-position (contd)

(if part) To prove:
    $N(G) = 0$ implies P-position

If $N(G)=0$, then by the MEX rule none of the successors of G have $N(G')=0$.

By induction all of them are N-positions.
Therefore G is a P-position!

The Nimber Theorem

Theorem: Let A and B be two impartial normal games. Then:

$$N(A+B) = N(A) \oplus N(B)$$

Proof: Deferred for now.

The beauty of Nimbers is that they *completely capture* what you need to know about a game in order to add it to another game.

This often allows you to compute winning strategies, and can speed up game search.
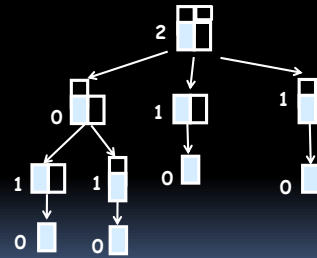
Application to Nim

Note that the game of Nim is just the sum of several games. If the piles are of size a, b, and c, then the nim game for these piles is just $P_a + P_b + P_c$.

The nimber of this position, by the Nimber Theorem, is just $a \oplus b \oplus c$.

So it's a P-position if and only if $a \oplus b \oplus c = 0$, which is what we proved before.

Application to Chomp

What is the nimber of this chomp game?

What if we add this to a nim pile of size 4?

$$4 \;\oplus\; 2 = 6$$

Is this an N-position or a P-position?

Nimber $\neq 0$. So it is an N-position. How do you win?

If we remove two chips from the nim pile, then the nimber is 0, giving a P-position. This is the unique winning move in this position.

The Game of Dawson's Kayles

Start with a row of n bowling pins:

A move consists of knocking down 2 neighboring pins.

The last player to move wins.

Note: an isolated pin is stuck and can never be removed.

How do we analyze this game?

Note that in a row of n pins there are $n-1$ possible moves [(x,y) denotes x pins to left and y pins to right of the two pins that are knocked down]

$$(0,n-2), (1,n-3), \dots , (n-3,1), (n-2,0)$$

So the nimber of a row of n pins, denoted N(n) is:

0 if n=0
0 if n=1
MEX{N(0)$\oplus$N(n-2), N(1)$\oplus$N(n-3), … N(n-2)$\oplus$N(0)}

Let's work out some small values…..

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
N(n)	0	0	1	1	2	0	3	1	1	0	3	3	2

The table has period 34.

Note that the case n=9 is an P-position

Proof of the Nimber Theorem:
$$N(A+B) = N(A) \oplus N(B)$$

Let the moves in A be $A_1, A_2, ..., A_n$
And the moves in B be $B_1, B_2, ..., B_m$

If either of these lists is empty, we just have one game, and the theorem is trivial.

The positions after a move in A+B are:
$A+B_1, A+B_2, ... A+B_m, A_1+B, ... A_n+B$

By definition
$N(A+B) = \text{MEX} \{ N(A+B_1),...N(A+B_m),$
$\qquad\qquad\qquad N(A_1+B),...N(A_n+B) \}$

So, by induction (what are we inducting on?),
$N(A+B) = \text{MEX}\{N(A)\oplus N(B_1),..., N(A)\oplus N(B_m),$
$\qquad\qquad\qquad N(A_1)\oplus N(B),..., N(A_n)\oplus N(B) \}$

How do we prove this is $N(A)\oplus N(B)$?
We do it by proving two things:
(1) $N(A)\oplus N(B)$ is not in the list
(2) $\forall\ y < N(A)\oplus N(B)$, y is in the list

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(1) $N(A)\oplus N(B)$ is not in the list

$\text{MEX}\{N(A)\oplus N(B_1),..., N(A)\oplus N(B_m),$
$\qquad\quad N(A_1)\oplus N(B),..., N(A_n)\oplus N(B) \}$

Why is $N(A)\oplus N(B)$ not in this list?

We have $N(B_i) \neq N(B)$ (Why?)
So $N(A)\oplus N(B_i) \neq N(A)\oplus N(B)$

Likewise $N(A_j) \neq N(A)$
Therefore $N(A_j)\oplus N(B) \neq N(A)\oplus N(B)$

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(2) For all $y < N(A)\oplus N(B)$, y is in the list

$N(A)\oplus N(B) = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ \boxed{1}\ 0\ 1\ 0\ 1\ 1\ 1$
$\qquad\qquad y = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ \boxed{0}\ .\ .\ .\ .\ .$
$\qquad\quad N(A) = .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \boxed{0}\ .\ .\ .\ .\ .$
$\qquad\quad N(B) = .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \boxed{1}\ .\ .\ .\ .\ .$

$N(B_i) = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ \boxed{0}\ x\ x\ x\ x\ x\ x$

The highlighted column is the first (from left to right) where y and $N(A)\oplus N(B)$ differ. At that bit position $N(A)\oplus N(B)$ is 1 and y is 0. So exactly one of N(A) and N(B) has a 1 in that position, WLOG assume it is N(B)

Because $N(B)=\text{MEX}\{N(B_1),...N(B_m)\}$, for every pattern of bits after the highlighted column, there is a B_i with that pattern in those columns (and 0 in highlighted column)
∴ we can produce the desired y by moving in B to some B_i.

Study Bee

• Combinatorial games
• Normal Versus Misère
• P-positions versus N-positions
• Mirroring
• Chomp
• Nim & Nim-sum
• Bouton's Theorem
• Sums of games
• Number of a game
• Nimber theorem

Supplementary material:
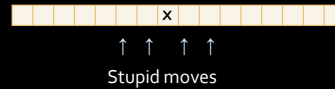Reducing one game to another

The Game of Treblecross

Tic-Tac-Toe on a line with only X's allowed.
First player to form 3-in-a-row wins.

Let's play. I go first:

First we eliminate "stupid" moves.
A stupid move is one which allows the opponent to win immediately on the next move.

↑ ↑   ↑ ↑
Stupid moves

Stupid move elimination does not change the outcome or the strategy of the game.

<u>Claim</u>: Treblecross of length n is *equivalent* to Dawson's Kayles[3] of length n+2.

(The [3] means you must take 3 pins in a row.)

<u>Proof idea</u> : Verify base cases (easy).

General case: Will argue that the game trees are identical.

Treblecross:

D. Kayles:

To the left of the X in the treblecross game, there is a treblecross game of size 5 (not counting stupid moves). This is equivalent (by induction) to the size 7 Dawson Kayles[3] game. The right side is the same.
The game trees are thus identical.

We can now evaluate the game (i.e., the nimbers of various positions) just as we did with regular Dawson's Kayles (that removed two pins)

n			0	1	2	3	4	5	6	7	8	9	10
n+2	0	1	2	3	4	5	6	7	8	9	10	11	12
N(n)	0	0	0	1	1	1	2	2	0	3	3	1	1