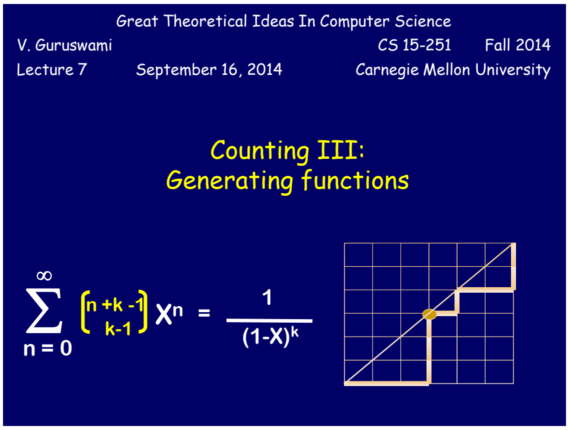
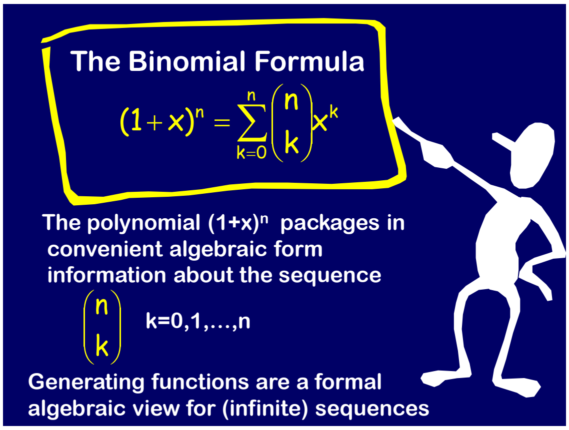


Great Theoretical Ideas In Computer Science
V. Guruswami CS 15-251 Fall 2014
Lecture 7 September 16, 2014 Carnegie Mellon University

Counting III:
Generating functions

$$\sum_{n=0}^{\infty} \left\{\begin{matrix} n+k-1 \\ k-1 \end{matrix}\right\} X^n = \frac{1}{(1-X)^k}$$



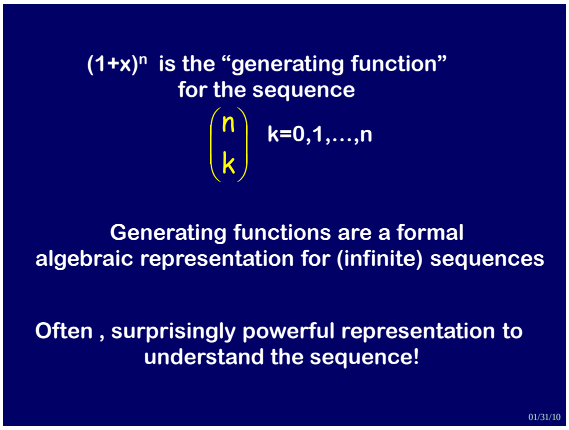
The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

The polynomial $(1+x)^n$ packages in convenient algebraic form information about the sequence

$\binom{n}{k}$ k=0,1,…,n

Generating functions are a formal algebraic view for (infinite) sequences



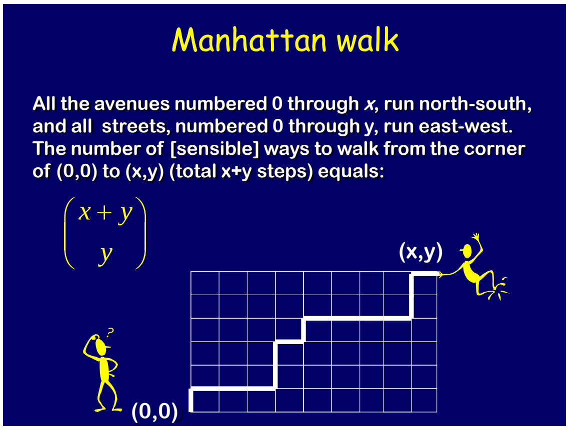
$(1+x)^n$ is the “generating function” for the sequence

$\binom{n}{k}$ k=0,1,…,n

Generating functions are a formal algebraic representation for (infinite) sequences

Often , surprisingly powerful representation to understand the sequence!

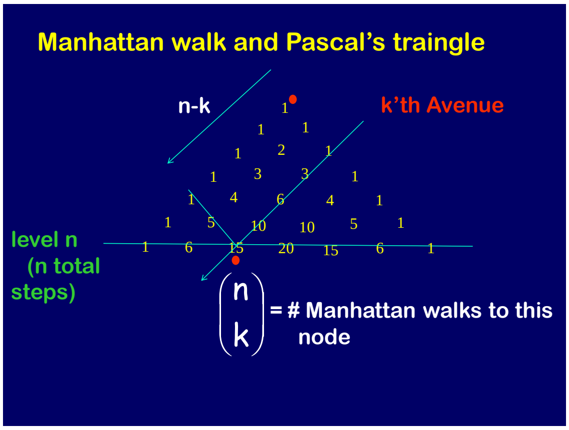
Manhattan Walks
Brief Recap



Manhattan walk

All the avenues numbered 0 through *x*, run north-south, and all streets, numbered 0 through y, run east-west. The number of [sensible] ways to walk from the corner of (0,0) to (x,y) (total x+y steps) equals:

$$\binom{x+y}{y}$$



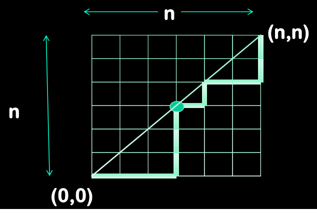
Manhattan walk and Pascal’s traingle

$\binom{n}{k}$ = # Manhattan walks to this node

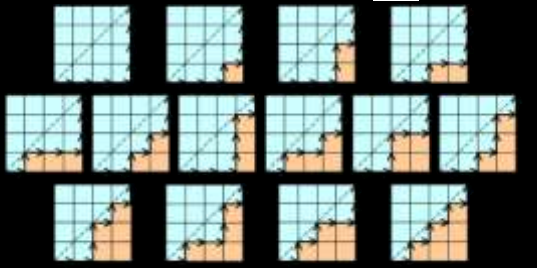
Noncrossing Manhattan walk

What if we require the Manhattan walk to **never cross the diagonal**?

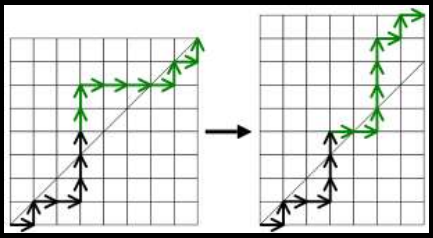
How many ways can we walk from (0,0) to (n,n) along the grid subject to this rule?



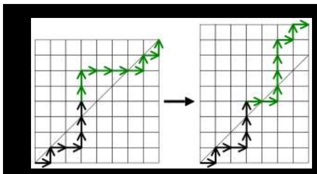
14 such walks for n=4
(c.f. total # Manhattan walks = $\binom{8}{4}$ = 70)



Let's count # violating paths, that *do* cross the diagonal
Will do so by a bijection.



Find first step above the diagonal.
"Flip" the portion of the path *after* that step.



Flip the portion of the path *after* the first edge above the diagonal.

Note: New path goes to (n-1,n+1)

Claim: The above is a ***bijection*** from crossing Manhattan walks in n x n grid to (unconstrained) Manhattan walks in (n-1,n+1) grid

Thus, number of noncrossing Manhattan walks on n x n grid = $\binom{2n}{n} - \binom{2n}{n-1}$

How many sequences of balanced paranthesis with n ('s and n 1)'s are there?

Answer: $c_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$

c_n is the **n'th *Catalan number.***

Feature Presentation

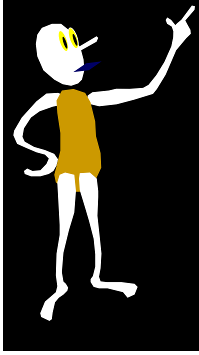
Generating Functions

Today we hope to answer:

What is a generating function, and why it is a powerful tool in one’s counting arsenal.

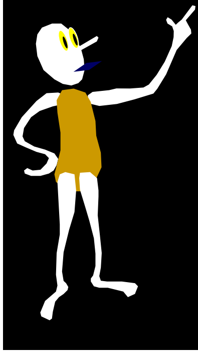
$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

$$= \frac{1 - X^n}{1 - X}$$



Recall the Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$



the <u>Infinite</u> Geometric Series

Holds when $|X| < 1$

Also makes sense if we view the infinite sum on the left as a formal power series

$$\begin{aligned} P(X) &= 1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots \\ -X * P(X) &= - X^1 - X^2 - X^3 - \ldots - X^n - X^{n+1} - \ldots \\ \hline (1-X)\,P(X) &= 1 \end{aligned}$$

$$\Rightarrow P(X) = \frac{1}{1 - X}$$

What is a Generating Function?

Just a particular representation of sequences… $\langle 1, 1, 1, \ldots \rangle$

$$1 + 1x + 1x^2 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

In general, when a_n is a sequence…

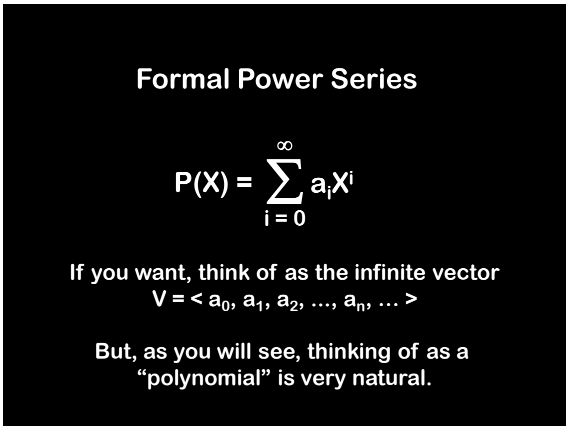
$$\sum_{n=0}^{\infty} a_n x^n$$

Formal Power Series

$$P(X) = \sum_{n=0}^{\infty} a_n X^n$$

There are no worries about convergence issues.

This is a purely syntactic object.

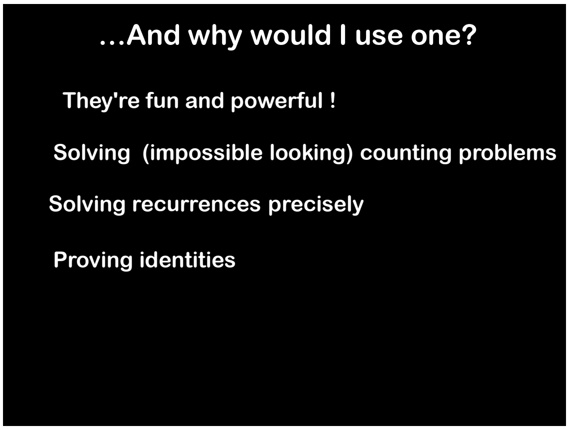


Formal Power Series

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

If you want, think of as the infinite vector
$V = < a_0, a_1, a_2, \ldots, a_n, \ldots >$

But, as you will see, thinking of as a
"polynomial" is very natural.



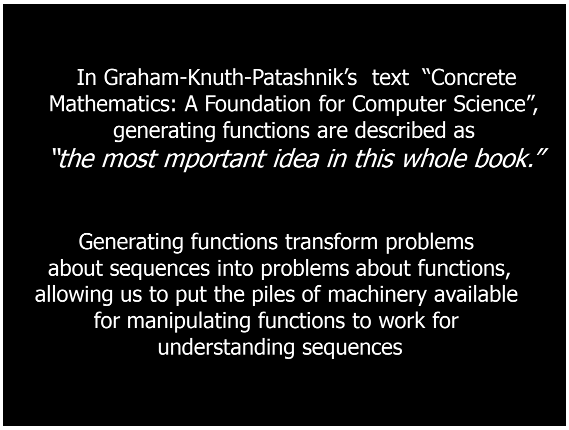
…And why would I use one?

They're fun and powerful !

Solving (impossible looking) counting problems

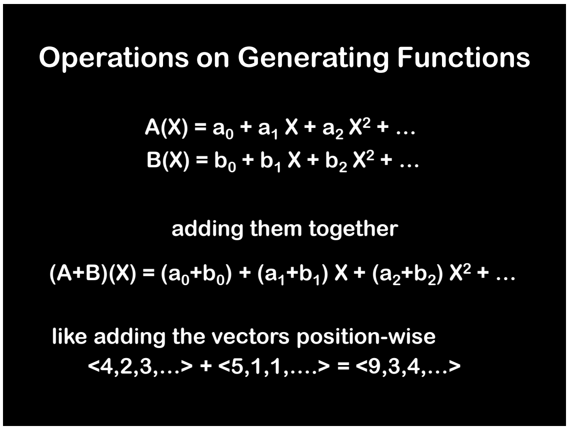
Solving recurrences precisely

Proving identities



In Graham-Knuth-Patashnik's text "Concrete Mathematics: A Foundation for Computer Science", generating functions are described as
"the most mportant idea in this whole book."

Generating functions transform problems about sequences into problems about functions, allowing us to put the piles of machinery available for manipulating functions to work for understanding sequences



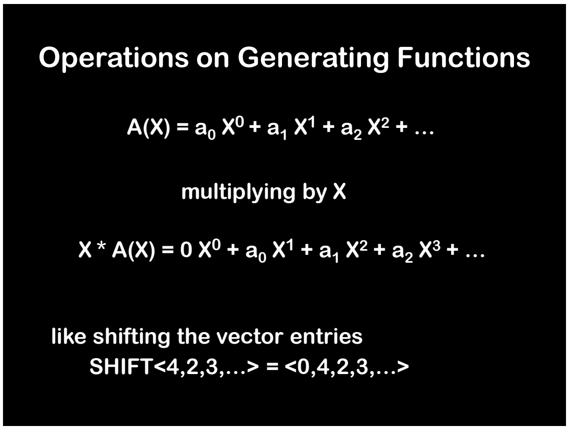
Operations on Generating Functions

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$
$B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$

adding them together

$(A+B)(X) = (a_0+b_0) + (a_1+b_1) X + (a_2+b_2) X^2 + \ldots$

like adding the vectors position-wise
<4,2,3,…> + <5,1,1,….> = <9,3,4,…>



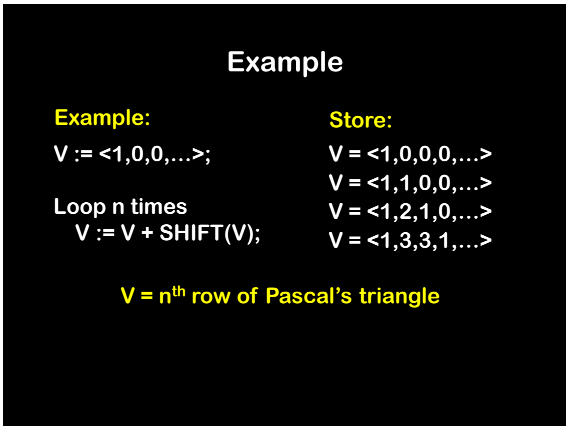
Operations on Generating Functions

$A(X) = a_0 X^0 + a_1 X^1 + a_2 X^2 + \ldots$

multiplying by X

$X * A(X) = 0 X^0 + a_0 X^1 + a_1 X^2 + a_2 X^3 + \ldots$

like shifting the vector entries
SHIFT<4,2,3,…> = <0,4,2,3,…>



Example

Example:

```
V := <1,0,0,…>;

Loop n times
   V := V + SHIFT(V);
```

Store:

V = <1,0,0,0,…>
V = <1,1,0,0,…>
V = <1,2,1,0,…>
V = <1,3,3,1,…>

V = n^{th} row of Pascal's triangle

Example

Example:

V := <1,0,0,…>; P_V := 1;

Loop n times
  V := V + SHIFT(V); $P_V := P_V*(1+X)$;

V = nth row of Pascal's triangle

Example

Example:

V := <1,0,0,…>;

Loop n times
  V := V + SHIFT(V);

$\}\ P_V = (1+X)^n$

V = nth row of Pascal's triangle

As expected, the coefficients of P_V give the nth row of Pascal's triangle

---

To repeat…

Given a sequence $V = < a_0, a_1, a_2, \dots, a_n, \dots >$

associate a formal power series with it

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

This is the "generating function" for V

Fibonaccis

$F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

i.e., the sequence <0,1,1,2,3,5,8,13…>

is represented by the power series

$0 + 1X^1 + 1X^2 + 2\,X^3 + 3\,X^4 + 5\,X^5 + 8\,X^6 + \dots$

Two Representations

$F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

$A(X) = 0 + 1X^1 + 1X^2 + 2\,X^3 + 3\,X^4 + 5\,X^5 + 8\,X^6 + \dots$

Can we write A(X) more succinctly?

$A(X) = F_0 + F_1\,X^1 + F2\,X^2 + F_3\,X^3 + \dots + F_n\,X^n + \dots$

$= X^1 + (F_1 + F_0)X^2 + (F_2+F_1)\,X^3 + \dots + (F_{n-1} + F_{n-2})\,X^n + ..$

$$= X + \sum_{m=1}^{\infty} F_m X^{m+1} + \sum_{m=0}^{\infty} F_m X^{m+2}$$

$$= X + X(A(X) - F_0) + X^2 A(X)$$

$$= X + XA(X) + X^2 A(X)$$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

G.F for Fibonaccis

$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$

has the generating function

$$A(X) = \frac{X}{(1 - X - X^2)}$$

i.e., the coefficient of X^n in $A(X)$ is F_n

$X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6$

$1 - X - X^2 \,\big)\, X$

$\underline{-(X - X^2 - X^3)}$

$X^2 + X^3$

$\underline{-(X^2 - X^3 - X^4)}$

$2X^3 + X^4$

$\underline{-(2X^3 - 2X^4 - 2X^5)}$

$3X^4 + 2X^5$

$\underline{-(3X^4 - 3X^5 - 3X^6)}$

$5X^5 + 3X^6$

$\underline{-(5X^5 - 5X^6 - 5X^7)}$

$8X^6 + 5X^7$

$-(8X^6 - 8X^7 - 8X^8)$

Two representations of the same thing…

$F_0 = 0,\ F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

Closed form expression for F_n?

$F_0 = 0,\ F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

let's factor $(1 - X - X^2)$

$$(1 - X - X^2) = (1 - \varphi_1 X)(1 - \varphi_2 X)$$

$$\text{where } \varphi_1 = \frac{1 + \sqrt{5}}{2}$$

$$\varphi_2 = \frac{1 - \sqrt{5}}{2}$$

Simplify, simplify…

$F_0 = 0,\ F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - \varphi_1 X)(1 - \varphi_2 X)}$$

some elementary algebra omitted…*

$$A(X) = \frac{1}{\sqrt{5}}\frac{1}{(1 - \varphi_1 X)} + \frac{-1}{\sqrt{5}}\frac{1}{(1 - \varphi_2 X)}$$

*you are not allowed to say this in _your_ answers…

$$A(X) = \frac{1}{\sqrt{5}}\frac{1}{(1 - \varphi_1 X)} + \frac{-1}{\sqrt{5}}\frac{1}{(1 - \varphi_2 X)}$$

$$\frac{1}{(1 - \varphi_1 X)} = 1 + \varphi_1 X + \varphi_1^2 X^2 + \ldots + \varphi_1^n X^n + \ldots$$

$$\frac{1}{1 - Y} = 1 + Y^1 + Y^2 + Y^3 + \ldots + Y^n + \ldots$$

the _Infinite_ Geometric Series

$$A(X) = \boxed{\frac{1}{\sqrt{5}}} \frac{1}{(1-\varphi_1 X)} \boxed{+} \boxed{\frac{-1}{\sqrt{5}}} \frac{1}{(1-\varphi_2 X)}$$

$$\frac{1}{(1-\varphi_1 X)} = 1 + \varphi_1 X + \varphi_1^2 X^2 + \ldots + \boxed{\varphi_1^n} X^n + \ldots$$

$$\frac{1}{(1-\varphi_2 X)} = 1 + \varphi_2 X + \ldots + \boxed{\varphi_2^n X^n} + \ldots$$

⇒ the coefficient of X^n in $A(X)$ is…

$$\frac{1}{\sqrt{5}} \varphi_1^n + \frac{-1}{\sqrt{5}} \varphi_2^n$$

Closed form for Fibonaccis

$$\boxed{F_n = \frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n}$$

where $\varphi = \frac{1 + \sqrt{5}}{2}$

"golden ratio"

Closed form for Fibonaccis

$$\boxed{F_n = \frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n}$$

F_n = closest integer to $\frac{1}{\sqrt{5}} \varphi^n$

To recap…

Given a sequence $V = < a_0, a_1, a_2, \ldots, a_n, \ldots >$

associate a formal power series with it

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

This is the "generating function" for V

We just used this for solving the Fibonacci recurrence…

Multiplication

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

$B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$

multiply them together

$$(A*B)(X) = (a_0 b_0) + (a_0 b_1 + a_1 b_0) X + (a_0 b_2 + a_1 b_1 + a_2 b_0) X^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) X^3 + \ldots$$

seems a bit less natural in the vector representation

(it's called a "convolution" there)

Multiplication: special case

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

Special case: $B(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$

multiply them together

$$(A*B)(X) = a_0 + (a_0 + a_1) X + (a_0 + a_1 + a_2) X^2 + (a_0 + a_1 + a_2 + a_3) X^3 + \ldots$$

it gives us partial sums!

For example…

Suppose $A(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$

and $B(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$

then $(A*B)(X) = 1 + 2X + 3X^2 + 4X^3 + \ldots$

$$= \frac{1}{1-X} * \frac{1}{1-X} = \frac{1}{(1-X)^2}$$

Generating function for integers <0,1,2,3,4…>

What happens if we again take prefix sums?

Take $1 + 2X + 3X^2 + 4X^3 + \ldots = \frac{1}{(1-X)^2}$

multiplying through by 1/(1-X)

$$\Delta_1 + \Delta_2 X^1 + \Delta_3 X^2 + \Delta_4 X^3 + \ldots = \frac{1}{(1-X)^3}$$

Generating function for the triangular numbers!

What's the pattern?

$\langle 1,1,1,1,\ldots \rangle = \frac{1}{1-X}$

$\langle 1,2,3,4,\ldots \rangle = \frac{1}{(1-X)^2}$

$\langle \Delta_1,\Delta_2,\Delta_3,\Delta_4,\ldots \rangle = \frac{1}{(1-X)^3}$

??? $= \frac{1}{(1-X)^k}$

What's the pattern?

$\binom{0}{0},\binom{1}{0},\binom{2}{0},\binom{3}{0},\ldots = \frac{1}{1-X}$

$\langle 1,2,3,4,\ldots \rangle = \frac{1}{(1-X)^2}$

$\langle \Delta_1,\Delta_2,\Delta_3,\Delta_4,\ldots \rangle = \frac{1}{(1-X)^3}$

??? $= \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0},\binom{1}{0},\binom{2}{0},\binom{3}{0},\ldots = \frac{1}{1-X}$

$\binom{1}{1},\binom{2}{1},\binom{3}{1},\binom{4}{1},\ldots = \frac{1}{(1-X)^2}$

$\langle \Delta_1,\Delta_2,\Delta_3,\Delta_4,\ldots \rangle = \frac{1}{(1-X)^3}$

??? $= \frac{1}{(1-X)^n}$

What's the pattern?

$\binom{0}{0},\binom{1}{0},\binom{2}{0},\binom{3}{0},\ldots = \frac{1}{1-X}$

$\binom{1}{1},\binom{2}{1},\binom{3}{1},\binom{4}{1},\ldots = \frac{1}{(1-X)^2}$

$\binom{2}{2},\binom{3}{2},\binom{4}{2},\binom{5}{2},\ldots = \frac{1}{(1-X)^3}$

??? $= \frac{1}{(1-X)^k}$

What's the pattern?

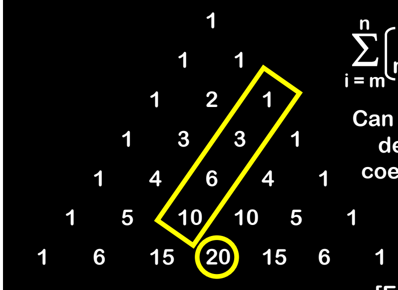
$$\binom{0}{0},\binom{1}{0},\binom{2}{0},\binom{3}{0},\ldots \quad = \frac{1}{1-X}$$

$$\binom{1}{1},\binom{2}{1},\binom{3}{1},\binom{4}{1},\ldots \quad = \frac{1}{(1-X)^2}$$

$$\binom{2}{2},\binom{3}{2},\binom{4}{2},\binom{5}{2},\ldots \quad = \frac{1}{(1-X)^3}$$

$$\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}X^n \quad = \frac{1}{(1-X)^k}$$

From last lecture: summing on avenues

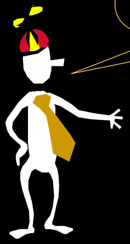


$$\sum_{i=m}^{n}\binom{i}{m}=\binom{n+1}{m+1}$$

Can be used to derive the coefficient of X^n in $\frac{1}{(1-X)^k}$

[Exercise!]

Another way to see it…



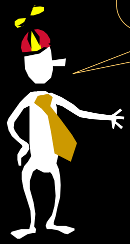
What is the coefficient of X^n in the expansion of:

$(1 + X + X^2 + X^3 + X^4 + \ldots)^k$?

Each path in the choice tree for the cross terms has n choices of exponent $e_1, e_2, \ldots, e_k \geq 0$. Each exponent can be any natural number.

Coefficient of X^n is the number of non-negative solutions to:
$e_1 + e_2 + \ldots + e_k = n$

Another way to see it…



What is the coefficient of X^n in the expansion of:

$(1 + X + X^2 + X^3 + X^4 + \ldots)^k$?

$$\binom{n+k-1}{k-1}$$

The Convolution Rule

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$ $B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$

GF for selecting items from set ***A*** GF for selecting items from set ***B***

A and ***B*** disjoint

Suppose there is a bijection between n-element selections from $A \cup B$ and ordered pairs of selections from ***A*** and ***B*** containing total of n els.

Then, number of ways to select n items total from $A \cup B = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots + a_nb_0$

GF for selecting items from disjoint union $A \cup B$
$= A(X)\,B(X)$

Another useful operation: Differentiation

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

differentiate it…

$A'(X) = a_1 + 2a_2 X + 3a_3 X^2 \ldots$

$$A'(X) = \sum_{i=0}^{\infty}(i+1)a_{i+1}X^i$$

$$X\,A'(X) = \sum_{i=0}^{\infty} i a_i X^i$$

Example of differentiation in action

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} X^n = \frac{1}{(1-X)^k}$$

$$\begin{aligned}\frac{1}{(1-X)^k} &= \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\left(\frac{1}{1-X}\right)\\ &= \frac{1}{(k-1)!}\sum_{\ell=(k-1)}^{\infty} \ell(\ell-1)\cdots(\ell-(k-2))\,X^{\ell-(k-1)}\\ &= \sum_{n=0}^{\infty}\frac{(n+k-1)(n+k-2)\cdots(n+1)}{(k-1)!}\,X^n\\ &= \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}X^n\,.\end{aligned}$$

Differentiation in use

Exercise: Prove that the generating function for squares, i.e.,
the sequence $a_n = n^2$, n=0,1,2…. equals

$$\frac{x(1+x)}{(1-x)^3}$$

Hint: Use differentiation + shifting twice

Integration

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

Integrating both sides ….

$$\int_0^X A(t)dt = a_0 X + a_1\frac{X^2}{2} + a_2\frac{X^3}{3} + \cdots$$

$$\frac{1}{X}\int_0^X A(t)dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}X^n$$

Example

Evaluate the sum $\sum_{i=0}^{n}\binom{n}{i}\frac{1}{(i+1)}$.

$$\sum_{i=0}^{n}\frac{\binom{n}{i}}{i+1}X^i = \frac{1}{X}\int_0^X (1+t)^n\,dt = \frac{(1+X)^{n+1}-1}{X(n+1)}$$

Substituting X=1, answer = $\frac{2^{n+1}-1}{n+1}$

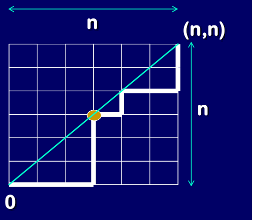
I like Catalan !

c_n = # ways to walk from (0,0) to (n,n) along the grid so that we never cross the diagonal

The bijection was clever but where did it come from?

A more systematic approach?

Recurrence + generating functions!



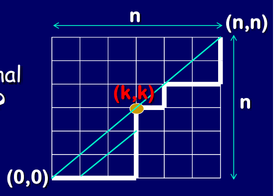
A recurrence

c_n = # Manhattan walksfrom (0,0) to (n,n) that never cross the diagonal (define c_0=1).

The walk must hit the diagonal at least once (perhaps only at the end).

walks that hit the diagonal at (k,k) for the *first* time? (1 ≤ k ≤ n)

Answer: c_{k-1} c_{n-k}



$$c_n = \sum_{k=1}^{n} c_{k-1}c_{n-k} = \sum_{i=0}^{n-1} c_i c_{n-1-i} \quad \text{for } n \geq 1$$

Catalan generating function

Define $C(x) = \sum_{n=0}^{\infty} c_n x^n$

Coefficient of x^{n-1} in $C(x)^2$

$$c_n = \sum_{k=1}^{n} c_{k-1}c_{n-k} = \sum_{i=0}^{n-1} c_i c_{n-1-i} \quad \text{for } n \geq 1$$

So $C(x) = x\,C(x)^2$ Hmm…

Be careful about c_0 (base cases)

Correct equation: $C(x) = 1 + x\,C(x)^2$

Catalan generating function

$C(x) = \sum_{n=0}^{\infty} c_n x^n$ $x\,C(x)^2 - C(x) + 1 = 0$

Solving the quadratic: $C(x) = \frac{1-\sqrt{1-4x}}{2x}$

Using this, one can calculate $c_n = \frac{1}{n+1}\binom{2n}{n}$

Define $D(x) = 2x\,C(x) = 1 - (1-4x)^{1/2} = \sum d_n x^n$

$d_n = \frac{D^{(n)}(0)}{n!} = \frac{2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-5)\cdot(2n-3)}{n!}$ $c_n = d_{n+1}/2$

Another take on Catalan GF

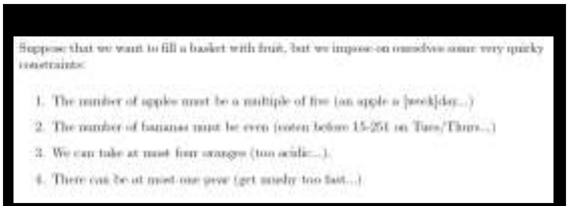
Let E(X) be the GF for *super non-crossing* Manhattan walks on n x n grids that *never touch the diagonal* (except at endpoints)

Fact 1: $E(X) = X\,C(X)$

Fact 2: $C(X) = 1 + E(X) + (E(X))^2 + (E(X))^3 + \ldots$

Together these imply $C(X) = \frac{1}{1 - XC(X)}$

Now to a seemingly over the top counting problem…



Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Let c_n = number of ways to pick exactly n fruits.

E.g., $c_5 = 6$

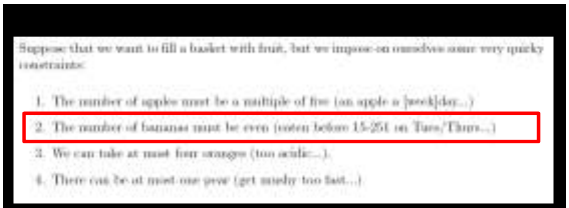
apples	0	0	0	0	0	5
bananas	4	4	2	2	0	0
oranges	1	0	2	3	4	0
pears	0	1	1	0	1	0

What is a closed form for c_n?

Recall Convolution Rule

So if A(x), B(x), O(x) and P(x) are the generating functions for the number of ways to fill baskets using only one kind of fruit

the generating function for number of ways to fill basket using any of these fruits is given by $C(x) = A(x)B(x)O(x)P(x)$



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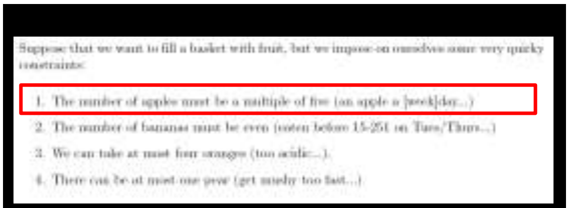
Suppose we only pick bananas

b_n = number of ways to pick n fruits, only bananas.

<1,0,1,0,1,0,...>

$$B(x) = 1 + x^2 + x^4 + x^6 + \dots \quad = \frac{1}{1-X^2}$$



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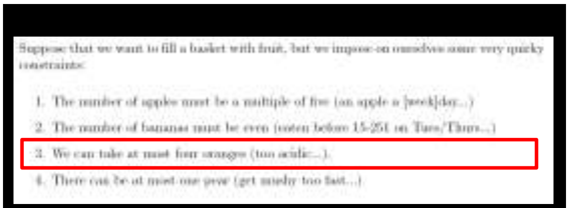
Suppose we only pick apples

a_n = number of ways to pick n fruits, only apples.

<1,0,0,0,0,1,...>

$$A(x) = 1 + x^5 + x^{10} + x^{15} + \dots \quad = \frac{1}{1-X^5}$$



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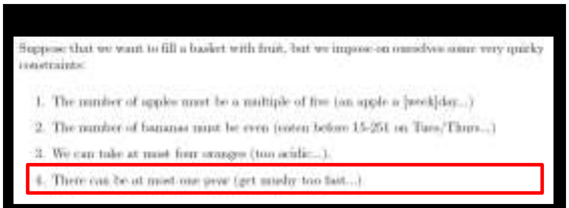
Suppose we only pick oranges

o_n = number of ways to pick n fruits, only oranges.

<1,1,1,1,1,0,0,0,...>

$$O(x) = 1 + x + x^2 + x^3 + x^4 \quad = \frac{1-X^5}{1-X}$$



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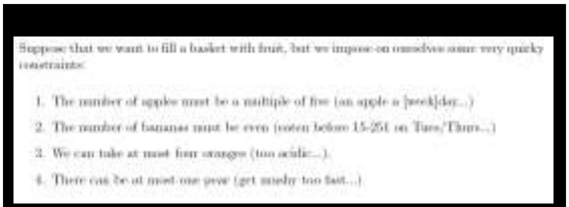
Suppose we only pick pears

p_n = number of ways to pick n fruits, only pears.

<1,1,0,0,0,0,0,...>

$$P(x) = 1 + x \quad = \frac{1-X^2}{1-X}$$



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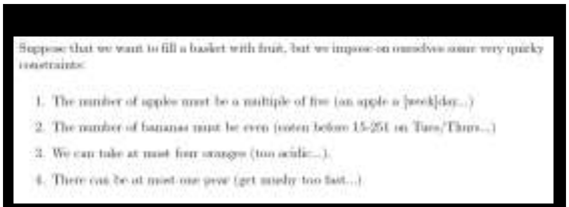
Let c_n = number of ways to
pick exactly n fruits of any type

$$\sum c_n x^n = A(x)\,B(x)\,O(x)\,P(x)$$

$$= \frac{1}{1-X^5}\;\frac{1}{1-X^2}\;\frac{1-X^5}{1-X}\;\frac{1-X^2}{1-X} \quad = \frac{1}{(1-X)^2}$$



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Let c_n = number of ways to
pick exactly n fruits of any type

c_n is coefficient of X^n in $\frac{1}{(1-X)^2}$

<1,2,3,4,...>

$$c_n = n+1.$$

Another recurrence example

$$d_n = 2d_{n-1} + 3d_{n-2} \qquad d_0 = 0 \quad d_1 = 1$$

Goal: derive a closed form using generating functions.

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n$

Proceeding as in Fibonacci example…

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = x + \sum_{n=2}^{\infty} (2d_{n-1} + 3d_{n-2})x^n$

$$= x + \sum_{n=2}^{\infty} 2d_{n-1}x^n + \sum_{n=2}^{\infty} 3d_{n-2}x^n$$

$$= x + 2x \sum_{n=2}^{\infty} d_{n-1}x^{n-1} + 3x^2 \sum_{n=2}^{\infty} d_{n-2}x^{n-2}$$

$$= x + 2x \sum_{n=1}^{\infty} d_n x^n + 3x^2 \sum_{n=0}^{\infty} d_n x^n$$

$$= x + 2x(D(x) - d_0) + 3x^2 D(x)$$

A closed form

$$D(x) = x + 2xD(x) + 3x^2 D(x)$$

$$(1 - 2x - 3x^2)D(x) = x$$

$$D(x) = \frac{x}{1 - 2x - 3x^2}$$

Simplifying to retrieve d_n

$$D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1 - 2x - 3x^2} = \frac{-1}{4(1+x)} + \frac{1}{4(1-3x)}$$

Factorize denominator to break it into smaller pieces!

$$\frac{x}{1 - 2x - 3x^2} = \frac{x}{(1+x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-3x}$$

$$x = (1-3x)A + (1+x)B$$

$$1 = -3A + B$$

$$0 = A + B$$

$$A = \frac{-1}{4}$$

$$B = \frac{1}{4}$$

Retrieving d_n

$$D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1 - 2x - 3x^2} = \frac{-1}{4(1+x)} + \frac{1}{4(1-3x)}$$

$$= \frac{-1}{4} \sum_{n=0}^{\infty} (-x)^n + \frac{1}{4} \sum_{n=0}^{\infty} (3x)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{4} \left((-1)^{n+1} + 3^n\right) x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1-(3x)} = \sum_{n=0}^{\infty} (3x)^n$$

$$d_n = \frac{1}{4} \left((-1)^{n+1} + 3^n\right)$$

Some Common GFs

Sequence	Generating Function
$\langle 1, 1, 1, \dots \rangle$	$\frac{1}{1-x}$
$\langle 1, 2, 4, \dots \rangle$	$\frac{1}{1-2x}$
$\langle 1, 2, 3, \dots \rangle$	$\frac{1}{(1-x)^2}$
$\langle 0, 1, 1, 2, 3, \dots \rangle$	$\frac{x}{1-x-x^2}$

Here's What You Need to Know…

- Formal Power Series
- Basic operations on Formal Power Series
- Solving recurrences using generating functions
  (handle base cases carefully!)
- Solving G.F. to get closed form
- G.F.s for common sequences
- Prefix sums using G.F.s
- Using G.F.s to solve counting problems