Great Theoretical Ideas In Computer Science

V. Guruswami — CS 15-251 — Fall 2014

Lecture 6 — September 11, 2014 — Carnegie Mellon University

COUNTING II: PIRATES, PIGEONS, PASCAL, CATALAN

Plan

Some recap

Pigeonhole Principle

Pascal's Triangle

Combinatorial Proofs

Manhattan Walk

Catalan Number

Permutations vs. Combinations

Subsets of r out of n distinct objects

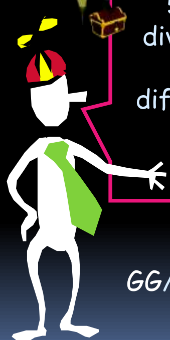
$$\frac{n!}{(n-r)!} = P(n,r) \qquad \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Ordered — Unordered

ways to arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

Multinomial Coefficient

$$\binom{n}{r_1;r_2;\ldots;r_k} = \begin{cases} 0, \text{if } r_1+r_2+\ldots+r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$



5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

GG/G/GG/GGGGGGGGGGGGGG/G

Sequences with 20 G's and 4 /'s

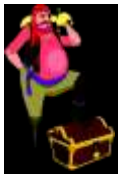
GG/G//GGGGGGGGGGGGGGG/G

In general, the j^{th} pirate gets the number of G's after the $j\text{-}1^{st}$ / and before the j^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

ways to divide up the loot
= # sequences with 20 G's and 4 /'s

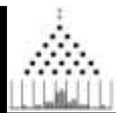
$$\binom{24}{4} = \binom{20+5-1}{5-1}$$



ways k ***distinct*** pirates can divide n ***identical***, indivisible bars of gold

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Another interpretation



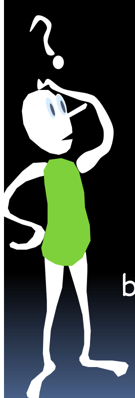
Number of different ways to throw n *indistinguishable* balls into k *distinguishable* bins:

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

How many nonnegative integer solutions to the following equations?

$$x_1 + x_2 + \ldots + x_k = n$$

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$



How many positive integer solutions to the following equations?

$x_1 + x_2 + x_3 + \ldots + x_k = n$

$x_1, x_2, x_3, \ldots, x_k > 0$

Think of $x_i \rightarrow y_i + 1$

bijection with solutions to

$y_1 + y_2 + y_3 + \ldots + y_k = n-k$ $\quad \binom{n-1}{k-1}$

$y_1, y_2, y_3, \ldots, y_k \geq 0$

Remember to distinguish between Identical / Distinct Objects

If we are putting n objects into k distinct bins.

n objects are distinguishable	k^n
n objects are indistinguishable	$\binom{n+k-1}{k-1}$

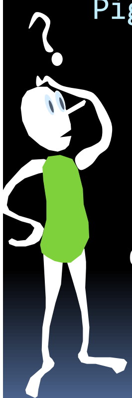
Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Pigeonhole Principle

If there are more pigeons than pigeonholes, then some pigeonholes must contain two or more pigeons

Example:
two people in Pittsburgh must have the same number of hairs on their heads



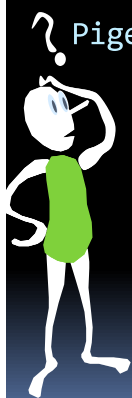
Pigeonhole Principle

Problem: Prove that if seven distinct numbers are selected from {1,2,3,…,11}, some two of these numbers sum up to 12.

Pigeons : the chosen numbers
Holes: Falling in same hole should mean numbers sum up to 12

6 holes: {1,11}, {2,10}, {3,9}, {4,8}, {5,7}, {6}
Place selected numbers in hole corresponding to the set containing it

Some two numbers fall in same hole, and thereby sum to 12.



Pigeonhole Principle

Problem:
Prove that among *any* n integers, there is a non-empty subset whose sum is divisible by n.

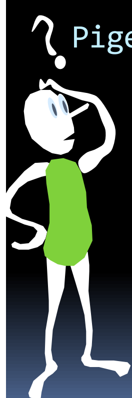
Among any n integer numbers, there is a non-empty subset whose sum is divisible by n.

Consider $s_i = x_1 + \ldots + x_i$ modulo n. How many s_i?
These are the n "pigeons"

Remainders modulo n belong to {0, 1, 2, …, n-1}.
If some remainder is 0, we are done.

If not, (n-1) remainders {1,2,…,n-1}. The "holes".

Exist s_i, s_k, $i < k$ such that n divides $s_k - s_i$.
$x_{i+1} + \ldots + x_k$ is our desired sum



Pigeonhole Principle

Problem:
The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17.

What are pigeons?
And what are pigeonholes?

The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17

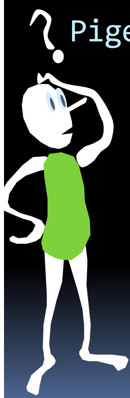
Let $S_1=a_1+a_2+a_3$, … $S_{10}=a_{10}+a_1+a_2$
There are 10 pigeonholes.

Pigeons: $S_1+ .. + S_{10} = 3\,(a_1+a_2+\ldots+a_{10}) = 3*55 = 165$

Since 165 > 10 *16, there must exist a pigeonhole with at least 16 + 1 pigeons

Actually, we're using a generalization of the PHP
If $x_1 + x_2 + \ldots + x_k > n$ then some $x_i > n/k$

Pigeonhole Principle



Problem:
Show that for some integer k > 1, 3^k ends with 0001 (in its decimal representation).

What are pigeons?
And what are pigeonholes?

Show that for some integer k > 1, 3^k ends with 0001

Choose 10001 numbers: $3^1, 3^2, \ldots, 3^{10001}$

$3^k = 3^m \bmod (10000)$, m < k

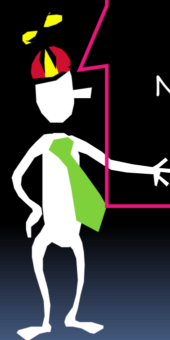
$3^k - 3^m = 0 \bmod (10000)$, m < k

$3^m(3^{k-m} - 1) = 0 \bmod (10000)$, m < k

But 3 is relatively prime to 10000, so

$3^{k-m} = 1 \bmod (10000)$

$3^{k-m} = q*10000 + 1$ ends with 0001

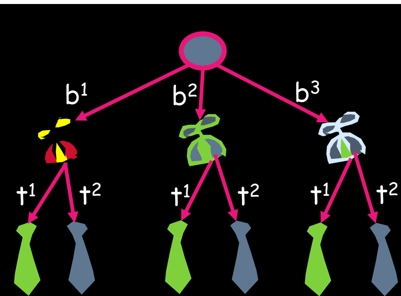


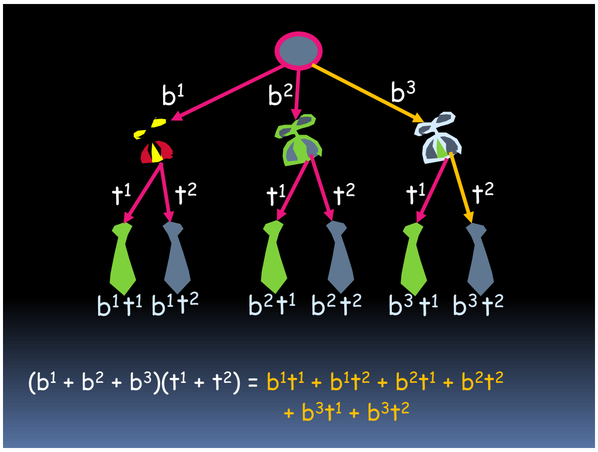
Now, to binomial theorem…

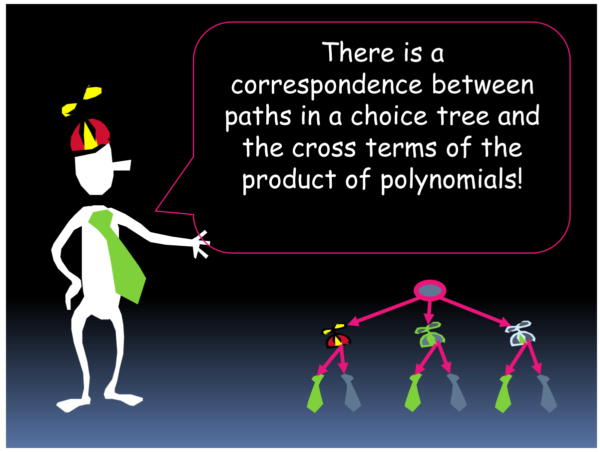
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

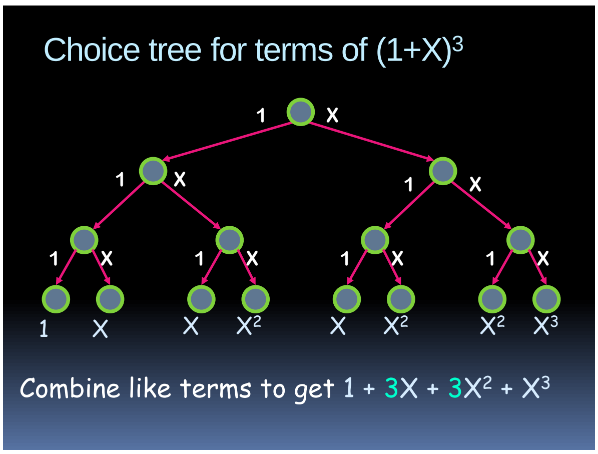
POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

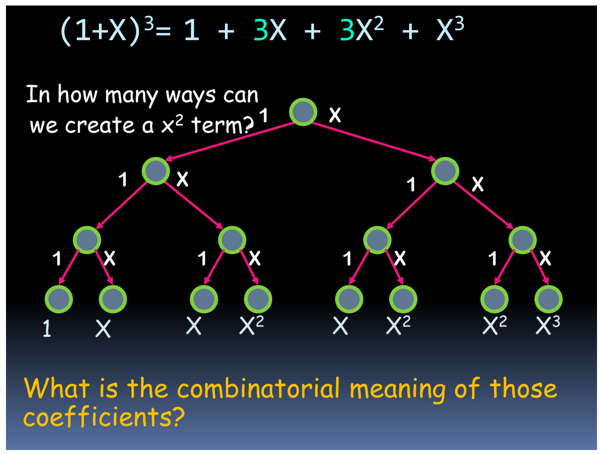
Products of Sum = Sums of Products

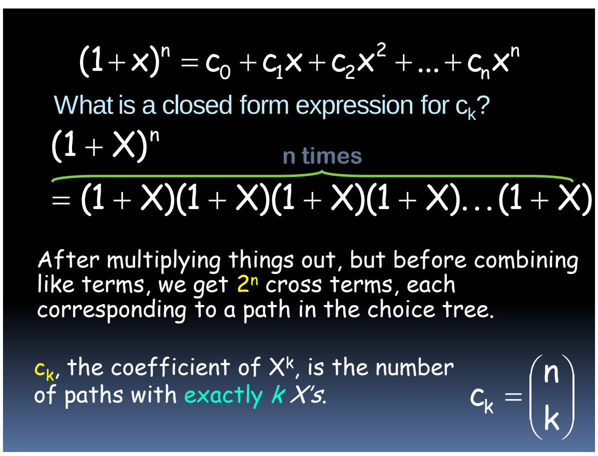


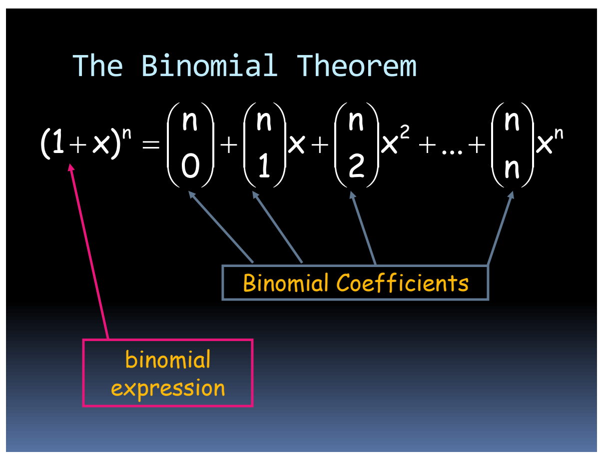

b^1 b^2 b^3
t^1 t^2 t^1 t^2 t^1 t^2
b^1t^1 b^1t^2 b^2t^1 b^2t^2 b^3t^1 b^3t^2
$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$

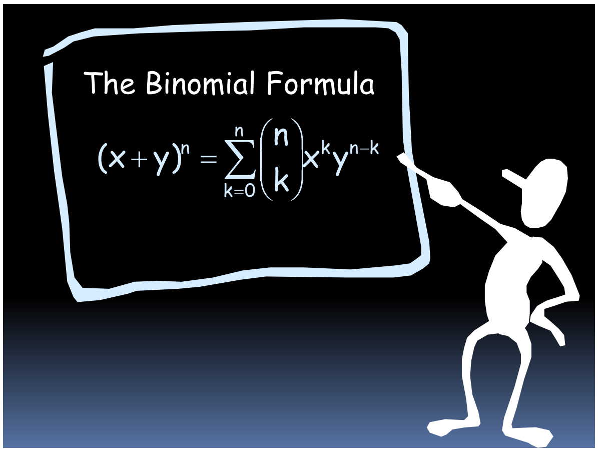

There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

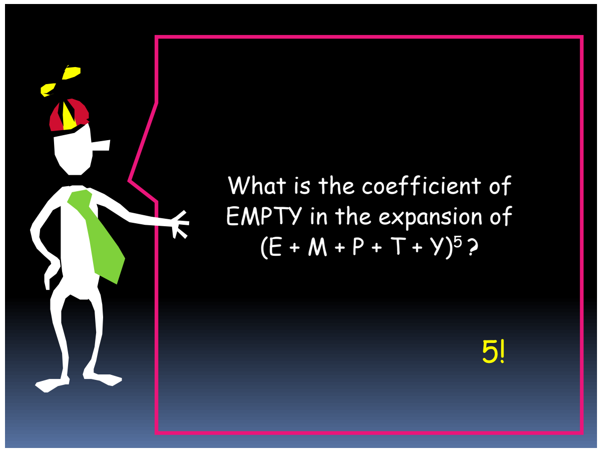

Choice tree for terms of $(1+X)^3$
1 X
1 X 1 X
1 X 1 X 1 X 1 X
1 X X X^2 X X^2 X^2 X^3
Combine like terms to get $1 + 3X + 3X^2 + X^3$

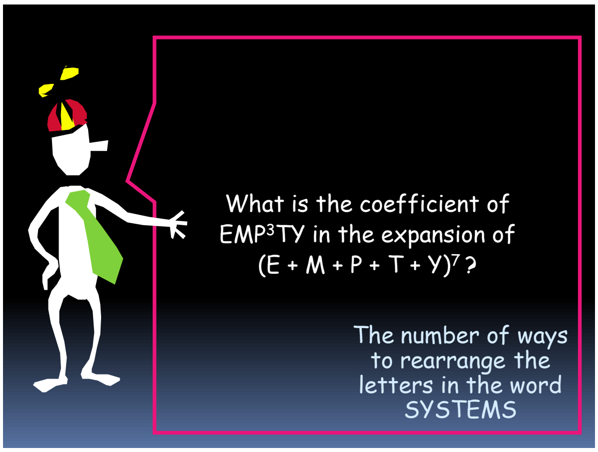

$(1+X)^3 = 1 + 3X + 3X^2 + X^3$
In how many ways can we create a x^2 term?
1 X
1 X 1 X
1 X 1 X 1 X 1 X
1 X X X^2 X X^2 X^2 X^3
What is the combinatorial meaning of those coefficients?

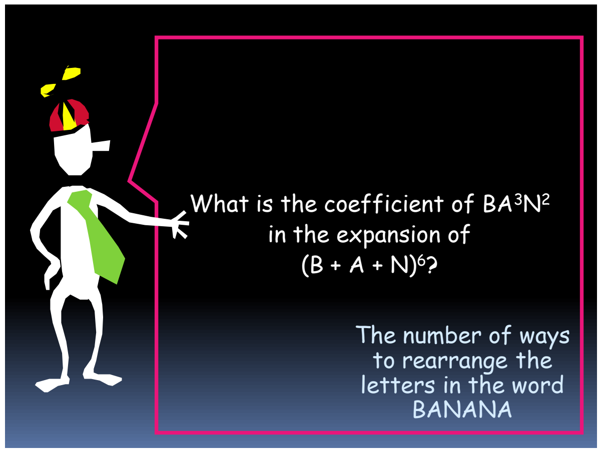

$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$
What is a closed form expression for c_k?
$(1+X)^n$
n times
$= (1+X)(1+X)(1+X)(1+X)\ldots(1+X)$
After multiplying things out, but before combining like terms, we get 2^n cross terms, each corresponding to a path in the choice tree.
c_k, the coefficient of X^k, is the number of paths with exactly k X's.
$c_k = \binom{n}{k}$

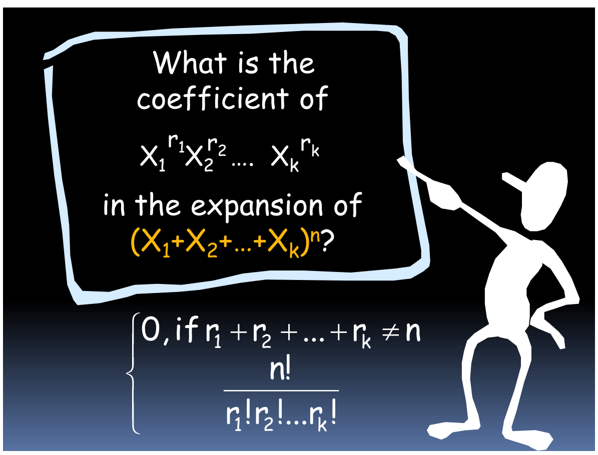

The Binomial Theorem
$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$
Binomial Coefficients
binomial expression

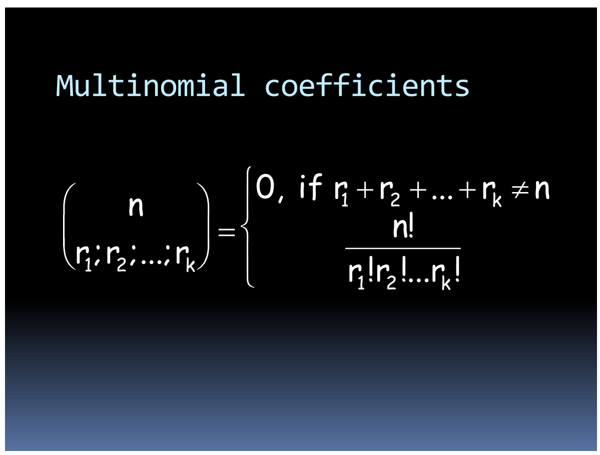

The Binomial Formula
$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$


What is the coefficient of EMPTY in the expansion of $(E + M + P + T + Y)^5$?
5!


What is the coefficient of EMP^3TY in the expansion of $(E + M + P + T + Y)^7$?
The number of ways to rearrange the letters in the word SYSTEMS


What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?
The number of ways to rearrange the letters in the word BANANA


What is the coefficient of
$X_1^{r_1} X_2^{r_2} \dots X_k^{r_k}$
in the expansion of
$(X_1+X_2+\dots+X_k)^n$?
$\begin{cases} 0, \text{if } r_1 + r_2 + \dots + r_k \neq n \\ \dfrac{n!}{r_1! r_2! \dots r_k!} \end{cases}$

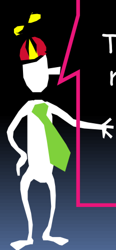

Multinomial coefficients
$\binom{n}{r_1; r_2; \dots; r_k} = \begin{cases} 0, \text{ if } r_1 + r_2 + \dots + r_k \neq n \\ \dfrac{n!}{r_1! r_2! \dots r_k!} \end{cases}$

The Multinomial Formula

$$(X_1+X_2+\dots+X_k)^n = \sum_{\substack{r_1,r_2,\dots,r_k \\ \sum r_i=n}} \binom{n}{r_1;r_2;\dots;r_k} X_1^{r_1}X_2^{r_2}X_3^{r_3}\dots X_k^{r_k}$$

On to Pascal…

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

Pascal's Triangle:
n^{th} row are the coefficients of $(1+X)^n$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

n^{th} Row Of Pascal's Triangle:

$$\binom{n}{0},\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n}$$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle

Blaise Pascal
1654

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1

1 + 1

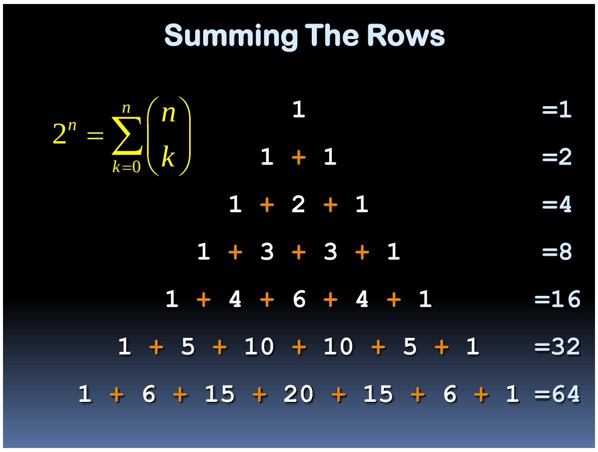
1 + 2 + 1

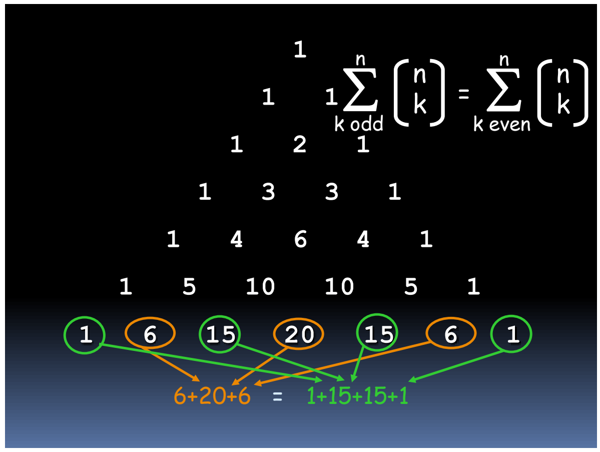
1 + 3 + 3 + 1

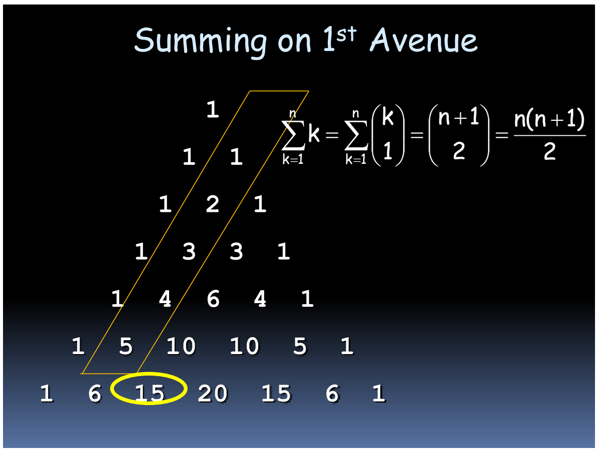
1 + 4 + 6 + 4 + 1

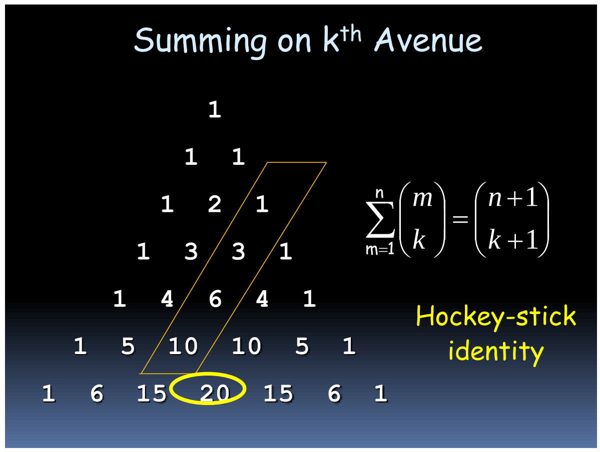
1 + 5 + 10 + 10 + 5 + 1

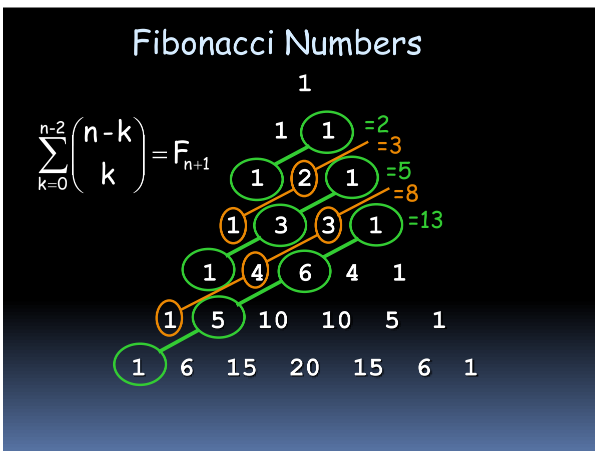
1 + 6 + 15 + 20 + 15 + 6 + 1

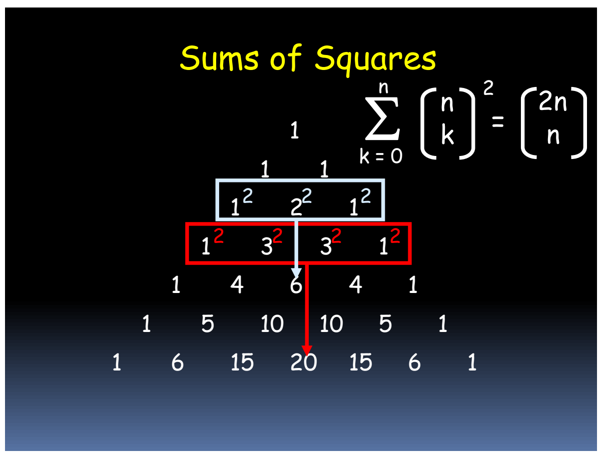

Summing The Rows
$2^n = \sum_{k=0}^{n} \binom{n}{k}$
1 =1
1 + 1 =2
1 + 2 + 1 =4
1 + 3 + 3 + 1 =8
1 + 4 + 6 + 4 + 1 =16
1 + 5 + 10 + 10 + 5 + 1 =32
1 + 6 + 15 + 20 + 15 + 6 + 1 =64


$\sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}$
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
6+20+6 = 1+15+15+1

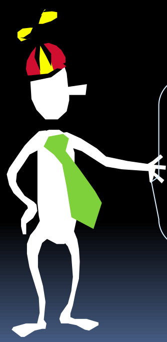

Summing on 1st Avenue
$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2} = \frac{n(n+1)}{2}$
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1


Summing on kth Avenue
$\sum_{m=1}^{n} \binom{m}{k} = \binom{n+1}{k+1}$
Hockey-stick identity
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1


Fibonacci Numbers
$\sum_{k=0}^{n-2} \binom{n-k}{k} = F_{n+1}$
1
1 1 =2
=3
1 2 1 =5
=8
1 3 3 1 =13
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1


Sums of Squares
$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
1
1 1
1^2 2^2 1^2
1^2 3^2 3^2 1^2
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The art of combinatorial proof



All these properties can be proved inductively and algebraically.
But we will give combinatorial proofs.

The art of combinatorial proof

$$\binom{n}{k} = \binom{n}{n-k}$$

How many ways we can create a size k committee out of n people?

LHS : By definition

RHS : We choose n-k people to exclude from the committee.

The art of combinatorial proof

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's identity

How many ways we can create a size k committee out of n people?

LHS : By definition

RHS : Pick a person, say n.

There are committees that exclude person x

There are committees that include person x

The art of combinatorial proof

$$\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$$

How many ways we can create an even size committee out of n people?

LHS : There are so many such committees

RHS : Choose an arbitrary subset of the first n-1 people. The fate of the nth person is completely determined.

The art of combinatorial proof

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

LHS : We create a size k committee, then we select a chairperson.

RHS : We select the chair out of n, then from the remaining n-1 choose a size k-1 committee.

The art of combinatorial proof

$$\sum_{k=0}^{n} k\binom{n}{k} = n\,2^{n-1}$$

LHS : Count committees of any size, one is a chair.

RHS : Select the chair out of n, then from the remaining n-1 choose a subset.

The art of combinatorial proof

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

Vandermonde's identity

LHS : m-males, n-females, choose size k.

RHS : Select a committee with j men, the remaining k-j members are women.

The art of combinatorial proof

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}$$

Hockeystick identity

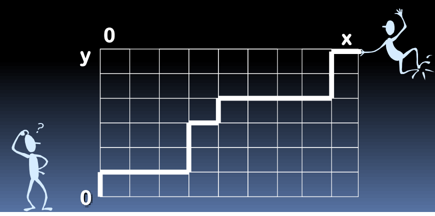
LHS : The number of (k+1)-subsets in {1,2,…,n+1}

RHS : Count (k+1)-subsets with the largest element m+1, where k ≤ m ≤ n.

All these properties can be proved by using the Manhattan walking representation of binomial coefficients.

Manhattan walk

You're in a city where all the streets, numbered 0 through *x*, run north-south, and all the avenues, numbered 0 through y, run east-west. How many [sensible] ways are there to walk from the corner of 0th st. and 0th avenue to the opposite corner of the city?

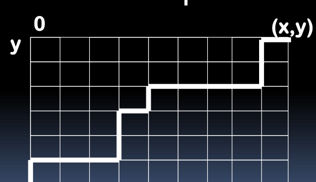


Manhattan walk

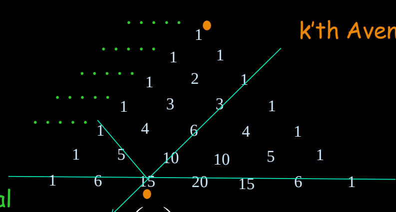
All paths require exactly x+y steps:
x steps east, y steps north

Counting paths is the same as counting which of the x+y steps are northward steps.

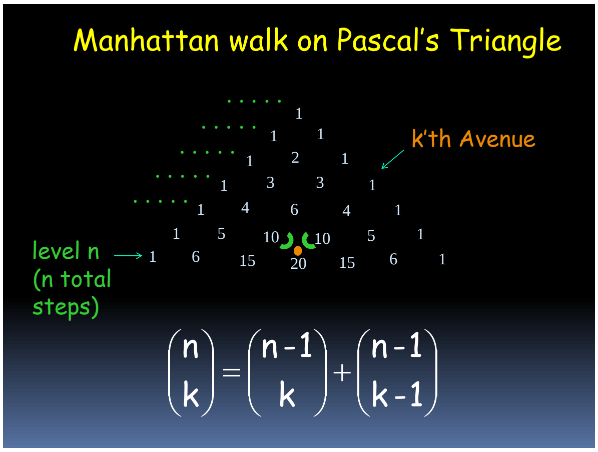
$$\binom{x+y}{y}$$

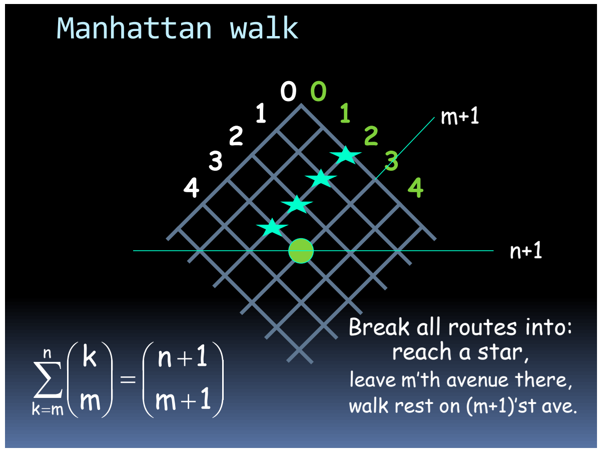


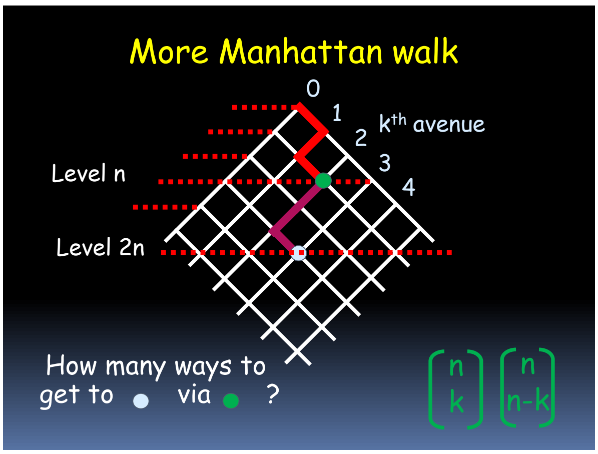
Manhattan walk on Pascal's triangle

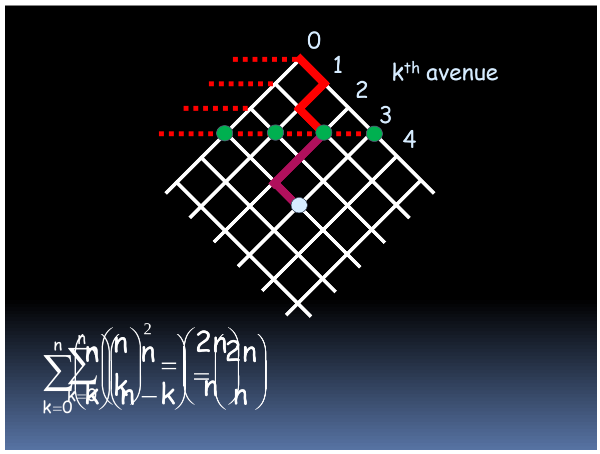


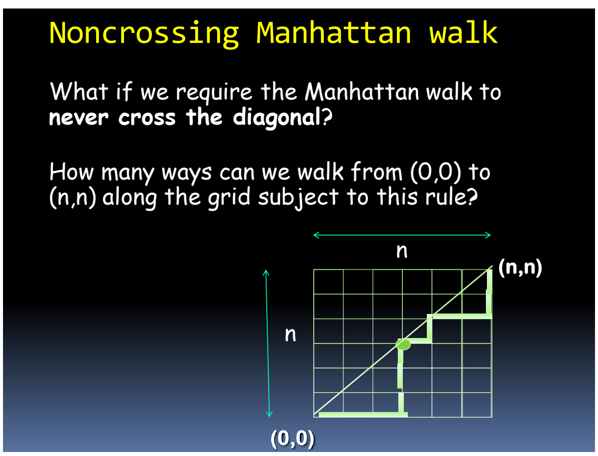
$\binom{n}{k}$ = # Manhattan walks to this node

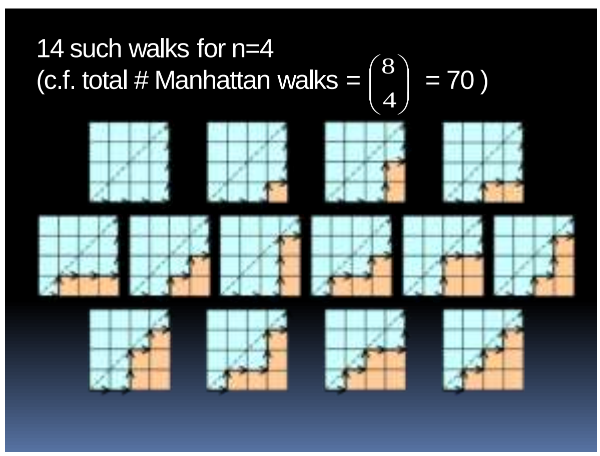

Manhattan walk on Pascal's Triangle
k'th Avenue
level n (n total steps)
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$


Manhattan walk
m+1
n+1
Break all routes into: reach a star, leave m'th avenue there, walk rest on (m+1)'st ave.
$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$

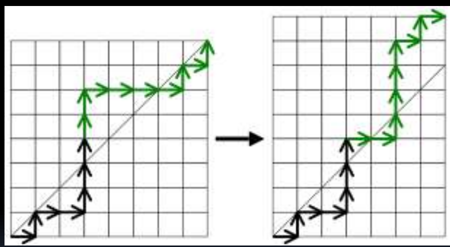

More Manhattan walk
k^{th} avenue
Level n
Level 2n
How many ways to get to ● via ● ?
$\binom{n}{k} \binom{n}{n-k}$


k^{th} avenue
$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

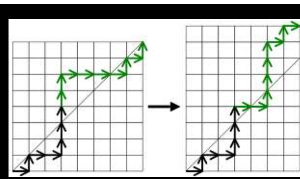

Noncrossing Manhattan walk
What if we require the Manhattan walk to **never cross the diagonal**?
How many ways can we walk from (0,0) to (n,n) along the grid subject to this rule?
n
(n,n)
n
(0,0)


14 such walks for n=4
(c.f. total # Manhattan walks = $\binom{8}{4}$ = 70)

Let's count # violating paths, that ***do*** cross the diagonal
Will do so by a bijection.



Find first step above the diagonal.
"Flip" the portion of the path ***after*** that step.



Flip the portion of the path ***after*** the first edge above the diagonal.

Note: New path goes to (n-1,n+1)

<u>Claim</u> (think about it):
Every Manhattan walk from (0,0) to (n-1,n+1) can be obtained in this fashion in *exactly one way*

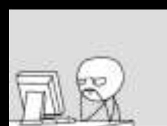
Thus, number of *noncrossing* Manhattan walks on n x n grid = $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$

How many sequences of n ('s and n)'s are there such that every prefix has more ('s than)'s?

Answer: $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$

The above is the **n'th** **Catalan number**.
Miraculously pervasive:

- # permutations of {1,2,…,n} that don't have 3 term increasing subsequence
- # ways the numbers 1, 2, ..., 2*n* can be arranged in a 2-by-*n* rectangle so that each row and each column is increasing.
- number of different ways a convex polygon with *n* + 2 sides can be cut into triangles by connecting vertices with straight lines



Study Guide

- Pirates and Gold
- Binomial & Multinomial theorems
- Pigeonhole principle
- Combinatorial proofs of binomial identities
- Manhattan walks
- Catalan numbers