15-251: Great Theoretical Ideas in Computer Science

Fall 2014, Lecture 5

Counting I

September 9, 2014

Straight Flush:	36
Four of a Kind:	624
Full House:	3,744
Flush:	5,112
Straight:	9,180
Three of a Kind:	54,912
Two Pair:	123,552
One Pair:	1,098,240
Nothing:	1,302,540
	2,598,960

In the next 3-4 lectures we will learn some fundamental counting methods.

Addition and Product Rules
The Principle of Inclusion-Exclusion
Choice Trees

Permutations and Combinations
The Binomial Theorem

The Pigeonhole Principle

Diophantine Equations

Recurrences.
Generating Functions

If I have 14 teeth on the top and 12 teeth on the bottom, how many teeth do I have in all?

Addition Rule

Let A and B be two disjoint finite sets

$$|A \cup B| = |A| + |B|$$

Addition of Multiple Disjoint Sets:

- Let $A_1, A_2, A_3, \ldots, A_n$ be disjoint, finite sets:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Addition Rule (2 Possibly Overlapping Sets)

Let A and B be two finite sets:

$$|A \cup B| =$$

$$|A| + |B| - |A \cap B|$$

Inclusion-Exclusion

- If A, B, C are three finite sets, what is the size of $(A \cup B \cup C)$?

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion

- If $A_1, A_2, \ldots, A_n$ are n finite sets, what is the size of $(A_1 \cup A_2 \cup \ldots \cup A_n)$?

$$\Sigma_i |A_i| - \Sigma_{i<j} |A_i \cap A_j| + \Sigma_{i<j<k} |A_i \cap A_j \cap A_k| \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Exercise: Prove this by induction!

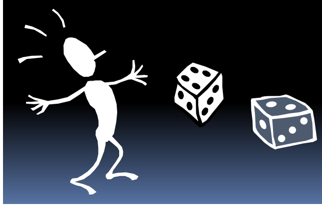
Partition Method

To count the elements of a finite set S, partition the elements into non-overlapping subsets $A_1, A_2, A_3, \ldots, A_n$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Partition Method

S = all possible outcomes of one white die and one black die.



Partition Method

S = all possible outcomes of one white die and one black die.

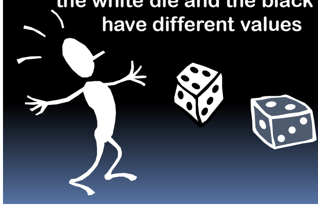
Partition S into 6 sets:

A_1 = the set of outcomes where the white die is 1.
A_2 = the set of outcomes where the white die is 2.
A_3 = the set of outcomes where the white die is 3.
A_4 = the set of outcomes where the white die is 4.
A_5 = the set of outcomes where the white die is 5.
A_6 = the set of outcomes where the white die is 6.

Each of 6 disjoint set have size 6 = 36 outcomes

Partition Method

S = all possible outcomes where the white die and the black die have different values



S ≡ Set of all outcomes where the dice show different values. | S | = ?

- A_i ≡ set of outcomes where black die says i and the white die says something else.

$$| S | = \sum_{i=1}^{6} | A_i | = \sum_{i=1}^{6} 5 = 30$$

S ≡ Set of all outcomes where the dice show different values. | S | = ?

B ≡ set of outcomes where dice agree.

$| S \cup B |$ = # of outcomes = 36

$|S| + |B| = 36$

$|B| = 6$

$|S| = 36 – 6 = 30$

Difference Method

To count the elements of a finite set S, find two sets A and B such that
S and B are disjoint
and
$S \cup B = A$

then $|S| = |A| - |B|$

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. | S | = ?

A_i ≡ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

S ≡ Set of all outcomes where the black die shows a smaller number than the white die. | S | = ?

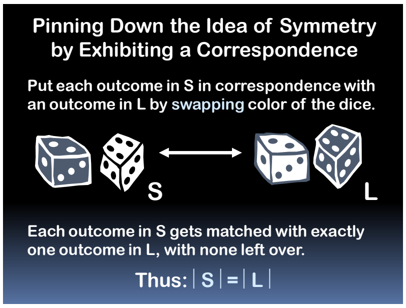
L ≡ set of all outcomes where the black die shows a larger number than the white die.

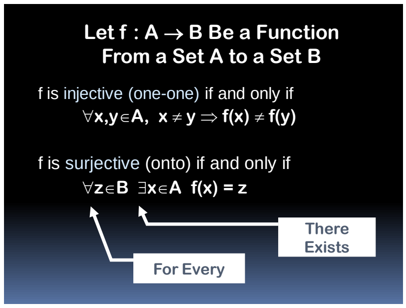
$$| S | + | L | = 30$$

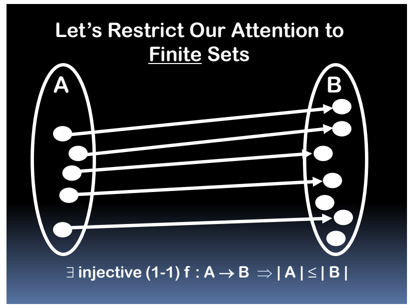
It is clear by symmetry that | S | = | L |.

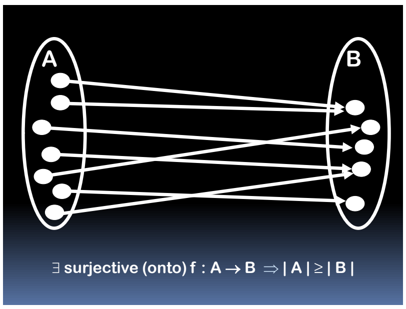
Therefore | S | = 15

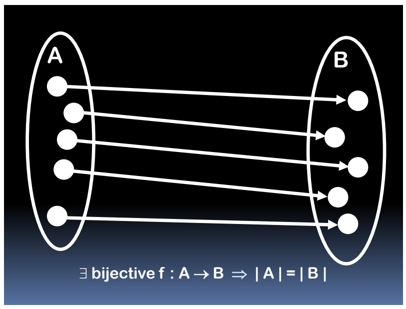
“It is clear by symmetry that |S| = |L|?”

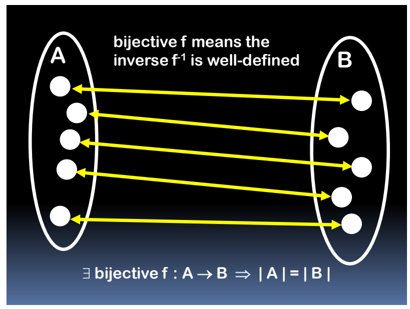

Pinning Down the Idea of Symmetry by Exhibiting a Correspondence
Put each outcome in S in correspondence with an outcome in L by swapping color of the dice.
S
L
Each outcome in S gets matched with exactly one outcome in L, with none left over.
Thus: | S | = | L |

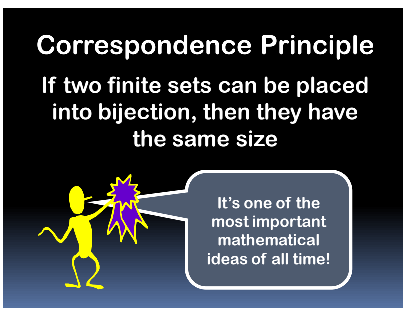

Let f : A → B Be a Function From a Set A to a Set B
f is injective (one-one) if and only if
∀x,y∈A, x ≠ y ⇒ f(x) ≠ f(y)
f is surjective (onto) if and only if
∀z∈B ∃x∈A f(x) = z
There Exists
For Every

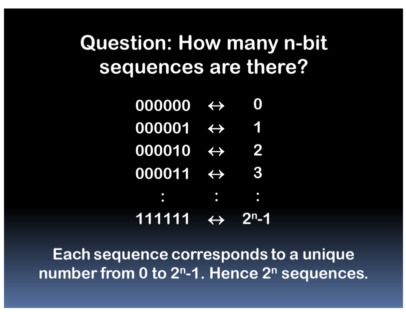

Let's Restrict Our Attention to Finite Sets
A
B
∃ injective (1-1) f : A → B ⇒ | A | ≤ | B |

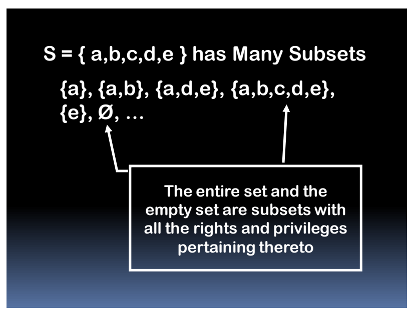

A
B
∃ surjective (onto) f : A → B ⇒ | A | ≥ | B |

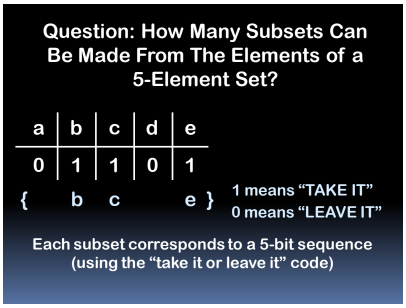

A
B
∃ bijective f : A → B ⇒ | A | = | B |

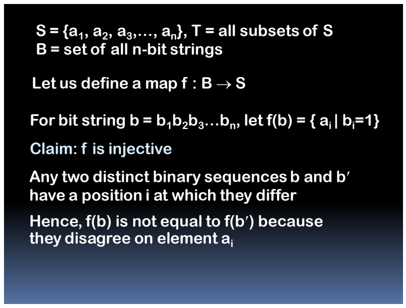

bijective f means the inverse f-1 is well-defined
A
B
∃ bijective f : A → B ⇒ | A | = | B |

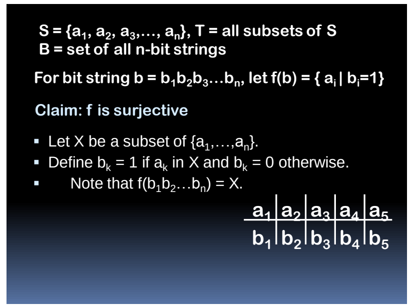

Correspondence Principle
If two finite sets can be placed into bijection, then they have the same size
It's one of the most important mathematical ideas of all time!

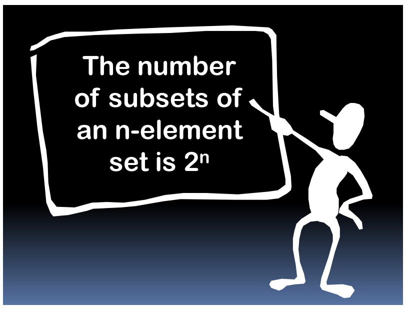

Question: How many n-bit sequences are there?
000000 ↔ 0
000001 ↔ 1
000010 ↔ 2
000011 ↔ 3
: : :
111111 ↔ 2n-1
Each sequence corresponds to a unique number from 0 to 2n-1. Hence 2n sequences.

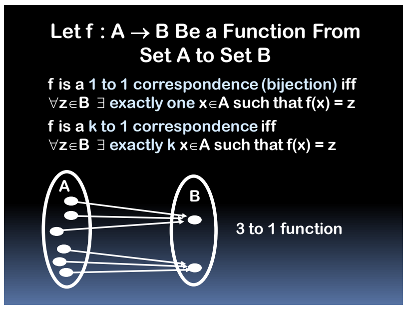

S = { a,b,c,d,e } has Many Subsets
{a}, {a,b}, {a,d,e}, {a,b,c,d,e}, {e}, Ø, …
The entire set and the empty set are subsets with all the rights and privileges pertaining thereto

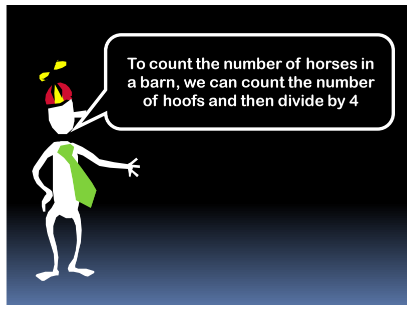

Question: How Many Subsets Can Be Made From The Elements of a 5-Element Set?
a b c d e
0 1 1 0 1
{ b c e }
1 means "TAKE IT"
0 means "LEAVE IT"
Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

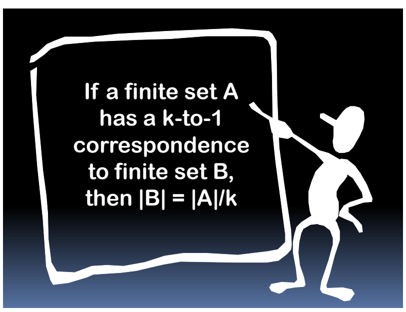

S = {a_1, a_2, a_3,…, a_n}, T = all subsets of S
B = set of all n-bit strings
Let us define a map f : B → S
For bit string b = $b_1b_2b_3…b_n$, let f(b) = { a_i | b_i=1}
Claim: f is injective
Any two distinct binary sequences b and b′ have a position i at which they differ
Hence, f(b) is not equal to f(b′) because they disagree on element a_i


S = {a_1, a_2, a_3,…, a_n}, T = all subsets of S
B = set of all n-bit strings
For bit string b = $b_1b_2b_3…b_n$, let f(b) = { a_i | b_i=1}
Claim: f is surjective
Let X be a subset of {a_1,…,a_n}.
Define b_k = 1 if a_k in X and b_k = 0 otherwise.
Note that f($b_1b_2…b_n$) = X.
a_1 a_2 a_3 a_4 a_5
b_1 b_2 b_3 b_4 b_5

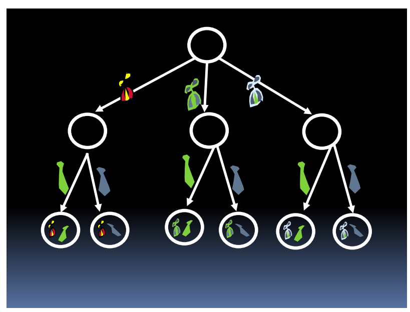

The number of subsets of an n-element set is 2^n


Let f : A → B Be a Function From Set A to Set B
f is a 1 to 1 correspondence (bijection) iff
∀z∈B ∃ exactly one x∈A such that f(x) = z
f is a k to 1 correspondence iff
∀z∈B ∃ exactly k x∈A such that f(x) = z
A
B
3 to 1 function


To count the number of horses in a barn, we can count the number of hoofs and then divide by 4


If a finite set A has a k-to-1 correspondence to finite set B, then |B| = |A|/k


I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?



A Restaurant Has a Menu With 5 Appetizers, 6 Entrees, 3 Salads, and 7 Desserts

How many items on the menu?

$5 + 6 + 3 + 7 = 21$

How many ways to choose a complete meal?

$5 \times 6 \times 3 \times 7 = 630$

How many ways to order a meal if I am allowed to skip some (or all) of the courses?

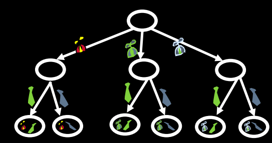
$6 \times 7 \times 4 \times 8 = 1344$

Leaf Counting Lemma

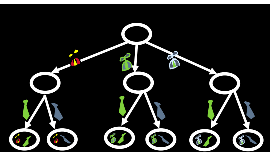
Let T be a depth-n tree when each node at depth $0 \leq i \leq n-1$ has P_{i+1} children

The number of leaves of T is given by: $P_1P_2\ldots P_n$

Choice Tree



A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf



A choice tree provides a "choice tree representation" of a set S, if

1. Each leaf label is in S, and each element of S is some leaf label
2. No two leaf labels are the same

We will now combine the correspondence principle with the leaf counting lemma to make a powerful counting rule for choice tree representation.

Product Rule

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1P_2P_3\ldots P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

: :

1 possible choice for the 52nd card.

By product rule: $52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52!$

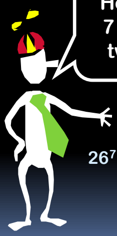
A permutation or arrangement of n objects is an ordering of the objects

The number of permutations of n distinct objects is n!

How many sequences of 7 letters are there?

26^7

(26 choices for each of the 7 positions)



How many sequences of 7 letters contain at least two of the same letter?

$26^7 - \underbrace{26\times25\times24\times23\times22\times21\times20}_{\text{number of sequences containing all different letters}}$

The Difference Principle

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.

If 10 horses race, how many orderings of the top three finishers are there?

$10 \times 9 \times 8 = 720$

Number of ways of ordering or arranging r out of n objects

n choices for first place, n-1 choices for second place, . . .

$n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$

$$= \frac{n!}{(n-r)!}$$

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$(52\times51) / 2$ ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52×51

How many unordered pairs?

$(52\times51) / 2$ ← divide by overcount

We have a 2-1 map from ordered pairs to unordered pairs.
Hence #unordered pairs = (#ordered pairs)/2

Ordered Versus Unordered

How many ordered 5 card sequences can be formed from a 52-card deck?

$52 \times 51 \times 50 \times 49 \times 48$

How many orderings of 5 cards?

5!

How many unordered 5 card hands?

$(52\times51\times50\times49\times48)/5! = 2{,}598{,}960$

A combination or choice of r out of n objects is an (unordered) set of r of the n objects

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n "choose" r

The number of subsets of size r that can be formed from an n-element set is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

How Many 8-Bit Sequences Have 2 0's and 6 1's?

Tempting, but incorrect:
8 ways to place first 0, times
7 ways to place second 0

Violates condition 2 of product rule! (uniqueness)

Choosing position i for the first 0 and then position j for the second 0 gives same sequence as choosing position j for the first 0 and then position i for the second 0 } 2 ways of generating the same object!

How Many 8-Bit Sequences Have 2 0's and 6 1's?

1. Choose the set of 2 positions to put the 0's. The 1's are forced. $\binom{8}{2}$

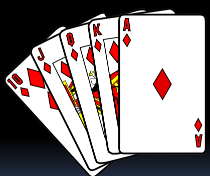
2. Choose the set of 6 positions to put the 1's. The 0's are forced. $\binom{8}{6}$

Symmetry In The Formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

"# of ways to pick r out of n elements"
=
"# of ways to choose the (n-r) elements to omit"

Counting Cards



How Many Hands Have at Least 3 As?

$\binom{4}{3}$ = ways of picking 3 out of 4 aces

$\binom{49}{2}$ = ways of picking 2 cards out of the remaining 49 cards

4 × 1176 = 4704

How Many Hands Have at Least 3 As?

How many hands have exactly 3 aces?

$\binom{4}{3}$ = ways of picking 3 out of 4 aces

$\binom{48}{2}$ = ways of picking 2 cards out of the 48 non-ace cards

$4 \times 1128 = 4512$

How many hands have exactly 4 aces?

$\binom{4}{4}$ = ways of picking 4 out of 4 aces

$\binom{48}{1}$ = ways of picking 1 cards out of the 48 non-ace cards

$+ 48$
4560

4704 ≠ 4560

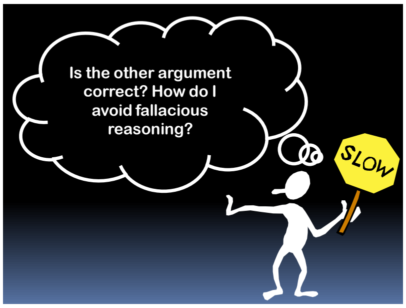
At least one of the two counting arguments is not correct!

Four Different Sequences of Choices Produce the Same Hand

$\binom{4}{3}$ = 4 ways of picking 3 out of 4 aces

$\binom{49}{2}$ = 1176 ways of picking 2 cards out of the remaining 49 cards

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦



REVERSIBILTY CHECK:
For each object can I reverse engineer the unique sequence of choices that constructed it?

Scheme I
1. Choose 3 of 4 aces
2. Choose 2 of the remaining cards

A♣ A♦ A♥A♠ K♦

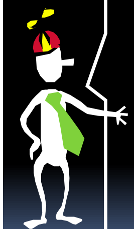
For this hand – you can't reverse to a unique choice sequence.

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A ♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Scheme II
1. Choose 3 out of 4 aces
2. Choose 2 out of 48 non-ace cards

A♣ A♦ Q♦ A♠ K♦

REVERSE TEST: Aces came from choices in (1) and others came from choices in (2)



The three big mistakes people make in associating a choice tree with a set S are:

1. Creating objects not in S
2. Missing out some objects from the set S
3. Creating the same object two different ways

DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I create every object of this type?

Can I reverse engineer my choice sequence from any given object?

A group of rabbits are playing outside their individual burrows when they are surprised by an eagle.

Each rabbit escapes down to a random hole, one rabbit per hole.

What is the chance that no rabbit is in its own individual hole?

How many ways are there for the rabbits to reorganize while avoiding their own hole?

A_i = set of permutations of six rabbits where i'th rabbit ends up in its hole

We want: $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5} \cap \overline{A_6}|$.

Of course: $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5} \cap \overline{A_6}| = 6! - |A_1 \cup A_2 \cup \cdots \cup A_6|$.

Use inclusion-exclusion.

$|A_i| = 5!$

$|A_i \cap A_j| = 4!$

$|A_i \cap A_j \cap A_k| = 3!$

$$|A_1 \cup \cdots \cup A_6| = 6 \cdot 5! - \binom{6}{2}4! + \binom{6}{3}3! - \binom{6}{4}2! + \binom{6}{5}1! - 1$$
$$= 6!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{6!}\right)$$

How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

_,_,_,_,_,_,_

7 places to put the Y,

6 places to put the T,
5 places to put the E,
4 places to put the M,

and the S's are forced

7 X 6 X 5 X 4 = 840

SYSTEMS

Let's pretend that the S's are distinct: $S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$

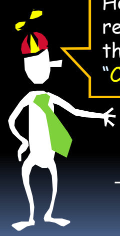
$$\frac{7!}{3!} = 840$$

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\ldots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!}\;\frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\;\cdots$$

$$= \frac{n!}{r_1!r_2!\ldots r_k!}$$



How many ways to rearrange the letters in the word "CARNEGIEMELLON"?

$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

Multinomial Coefficients

$$\binom{n}{r_1;r_2;\ldots;r_k} = \begin{cases} 0, \text{if } r_1+r_2+\ldots+r_k \neq n \\ \dfrac{n!}{r_1!r_2!\ldots r_k!} \end{cases}$$

Four ways of choosing

We will choose 2 leters from the alphabet {L,U,C,K,Y}

1) $\binom{5}{2}$ no repetitions, the order is NOT important LU = UL

Four ways of choosing

We will choose 2 letters from the alphabet {L,U,C,K,Y}

2) P(5,2) no repetitions, the order is important LU != UL

P(n,r)=n*(n-1)*…*(n-r+1)

Four ways of choosing

We will choose 2-letters word from the alphabet {L,U,C,K,Y}

3) 5^2 =25 with repetitions, the order is important

Four ways of choosing

We will choose 2-letter words from the alphabet {L,U,C,K,Y}

4) ???? Repetitions allowed, the order is NOT important

$\binom{5}{2}$ + |{LL,UU,CC,KK,YY}| = 15

What if we choose 3-letter words from the alphabet {L,U,C,K,Y}

allow repetitions, the order is NOT important

$\binom{5}{3} + 5 + \binom{5}{2} * 2 = 35$

What about 5-letter words? 20-letter words?

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

GG/G/GG/GGGGGGGGGGGGGG/G

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGG/G
represents the following division among the pirates

1st pirate gets 2
2nd pirate gets 1
3rd gets nothing (this is allowed!)
4th gets 16
5th gets 1

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGG/G

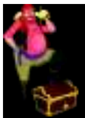
In general, the j^{th} pirate gets the number of G's after the j-1st / and before the j^{th} /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

How many sequences with 20 G's and 4 /'s?

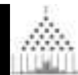
$$\binom{24}{4} = \binom{20+5-1}{5-1}$$



How many different ways can k *distinct* pirates divide n *identical*, indivisible bars of gold?

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Another interpretation



How many different ways to throw n *indistinguishable* balls into k *distinguishable* bins?

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Another interpretation

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

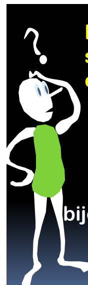
Think of x_k as being the number of gold bars that are allotted to pirate k.

$$\binom{24}{4}$$

How many integer nonnegative solutions to the following equations?

$$x_1 + x_2 + \ldots + x_r = m$$

$$\binom{m+r-1}{r-1} = \binom{m+r-1}{m}$$



How many integer positive solutions to the following equations?

$x_1 + x_2 + x_3 + \ldots + x_k = n$

$x_1, x_2, x_3, \ldots, x_k > 0$

Think of x_i -> y_i+1

bijection with solutions to

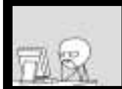
$y_1 + y_2 + y_3 + \ldots + y_k = n-k$ $\binom{n-1}{k-1}$

$y_1, y_2, y_3, \ldots, y_k \geq 0$

Remember to distinguish between Identical / Distinct Objects

If we are putting n objects into k distinct bins.

n objects are distinguishable	k^n
n objects are indistinguishable	$\binom{n+k-1}{k-1}$



Study guide

Partition and Difference Methods
- Principle of Inclusion and Excl.

Correspondence Principle
- If two finite sets can be placed into 1-1 onto correspondence, then they have the same size

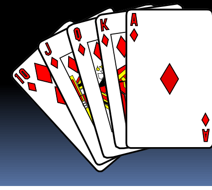
Choice Tree

Product Rule
- Two conditions

Reverse Test

Binomial & multinomial coefficient

Supplement: Testing your counting with Poker Hands



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52 Card Deck, 5 card hands

4 possible suits:
♥ ♦ ♣ ♠

13 possible ranks:
2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush: a straight and a flush

4 of a kind: 4 cards of the same rank

Full House: 3 of one kind and 2 of another

Flush: a flush, but not a straight

Straight: a straight, but not a flush

3 of a kind: 3 of the same rank, but not a full house or 4 of a kind

2 Pair: 2 pairs, but not 4 of a kind or a full house

A Pair

Straight Flush

9 choices for rank of lowest card at the start of the straight

4 possible suits for the flush

9 × 4 = 36

$\frac{36}{\binom{52}{5}} = \frac{36}{2,598,960}$ = about 1 in 72193 chance

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4 of a Kind

13 choices of rank

48 choices for remaining card

13 × 48 = 624

$\frac{624}{\binom{52}{5}} = \frac{624}{2,598,960}$ = 1 in 4165

Flush

4 choices of suit
$\binom{13}{5}$ choices of cards
} 4 × 1287 = 5148

"but not a straight flush…" - 36 straight flushes

5112 flushes

$\frac{5,112}{\binom{52}{5}}$ = 1 in 508.4…

Straight

9 choices of lowest card
4^5 choices of suits for 5 cards
} 9 × 1024 = 9216

"but not a straight flush…" - 36 straight flushes

9180 straights

$\frac{9,180}{\binom{52}{5}}$ = 1 in 208.1…

Hand	Number
Straight Flush:	36
Four of a Kind:	624
Full House:	3,744
Flush:	5,112
Straight:	9,180
Three of a Kind:	54,912
Two Pair:	123,552
One Pair:	1,098,240
Nothing:	1,302,540
	2,598,960