15-251 : Great Theoretical Ideas In Computer Science

Fall 2014

Assignment 10

Due: **Thursday**, November 20, 2014 11:59 PM

Name: ______________________________

Andrew ID: ______________________________

Question:	1	2	3	4	Total
Points:	30	20	20	30	100
Score:					

1. Proofs of 251 is composite

Let $L = \{x \in \{0,1\}^* \mid\ x \text{ encodes a proof, in ZFC, that 251 is composite}\}$.

Assume that ZFC is sound, and is capable of expressing any ordinary mathematical reasoning, for example, about Turing machines and proofs. Answer the following questions with clear justifications.

(4) (a) Is L the empty language?

Solution:

(6) (b) Is your answer to part (a) provable in ZFC?

Solution:

(4) (c) Is L decidable?

Solution:

(6) (d) Is your answer to (c) provable in ZFC?

Solution:

(10) (e) Describe a Turing machine M, such that the statement "M runs in at most $2^{O(n)}$ time" where n is the length of M's input, is neither provable nor disprovable in ZFC.

Solution:

2. Regular or Not, Take II

(20) (a) Prove that there is no Turing machine that halts on all inputs and correctly determines if an input string encodes a Turing machine that accepts a regular language. Formally, prove that

$$\mathrm{REG}_{\mathrm{T}M} = \{\langle M\rangle \mid L(M) \text{ is regular }\}$$

is undecidable.

Solution:

3. Turing times two

(20) (a) Define a Double Turing Machine as a Turing Machine with two tapes, and two heads. At each step, it can read from either tape, write to one or both tapes, and move neither, one, or both heads.

Given a Double Turing Machine, explain how it can be simulated using a normal Turing Machine as given in lecture. Thereby prove that a language L is decidable if and only if a Double Turing Machine halting on all inputs accepts exactly those strings in L.

Solution:

4. Recursively Enumerable Sets

Define a set $S \subseteq \Sigma^*$ to be recursively enumerable if there exists an algorithm such that the set of inputs for which the algorithm halts is exactly S.

(10) (a) Define a prime p to be a twin prime if both p and $p+2$ are prime. Prove that the set of twin primes, encoded as binary strings, is recursively enumerable.

Solution:

(10) (b) Prove that the halting set is recursively enumerable. In other words, prove that

$$\text{HALT} = \{\langle M, i\rangle \mid \text{ Turing Machine M halts on input i } \}$$

is recursively enumerable.

Solution:

(10) (c) Prove that the complement of the halting set is not recursively enumerable.

Solution: