

**15-251 : Great Theoretical Ideas In Computer Science****Fall 2014****Assignment 4**

Due: Thursday, Sep. 25, 2014 11:59 PM

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Question:	1	2	3	4	Total
Points:	30	20	15	35	100
Score:					

## 0. Warmup

(a) Find the coefficient of  $x^8$  in  $(1 + x^2 + x^4)(1 + x)^m$ .

(b) Find the coefficient of  $x^{18}$  in  $(1 + x^3 + x^6 + x^9 + \dots)^7$ .

(c) We noted in class that for the Fibonacci numbers, with  $f_0 = 0, f_1 = 1$ , the generating function of  $f_n$  is:

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{x}{1 - x - x^2}$$

Find the closed form of the generating function  $E(x)$  of the even Fibonacci numbers:

$$E(x) = \sum_{n=0}^{\infty} f_{2n} x^n$$

(d) Find the Nim-sum of 38 and 210.

(e) In Nim, is  $(13, 12, 8)$  a P-position? If not, what is a winning move?

(f) Two players play a game by starting with the integer 1000, and taking turns replacing the current integer  $N$  with either  $\lfloor \frac{N}{2} \rfloor$  or  $N - 1$ . The player that moves to 0 wins. Assuming optimal play, which player has a winning strategy?

(g) Find a generating function for the number of ways to make  $n$  cents using pennies, nickels, and dimes. The order of the coins does not matter.

**1. I am the master of generating functions**

- (10) (a) Find the generating function for the number Let  $a_n$  be the number of ways to partition a set of  $n$  elements into three unordered non-empty subsets. We have  $a_i = 0$  for  $i = 0, 1, 2$ ,  $a_3 = 1$ ,  $a_4 = 6$ ,  $a_5 = 25$ , etc. Find the generating function of the sequence  $a_n$ .

**Solution:**

- (10) (b) Find a generating function for the number of integer solutions of

$$2x + 3y + 7z = n$$

where

$$0 \leq z \leq 2 \leq y < 8 < x.$$

Explain your work.

**Solution:**

- (10) (c) Prove the following identity for positive integers  $n \geq k$ :

$$\sum_{\ell=0}^{3k} \sum_{j=0}^{\ell} (-1)^{\ell} \binom{n}{3k-\ell} \binom{n}{j} \binom{j}{\ell-j} = \binom{n}{k}.$$

**Solution:**

## 2. I am now a cupcakes master

Klaas, Peter, Patrick, Andy, David, and Taehoon each made their own type of awesome cupcakes. Klaas's had sprinkles, Peter's had rainbow icing, Patrick's had skittles, Andy had gummi bears, David's had drawings of ducks, and Taehoon's had those little umbrellas that fancy restaurants give you. Venkat and Victor worked together to make normal cupcakes. Venkat and Victor brought a  $1 \times n$  tray to put all the cupcakes in; the TAs decided that none of the special cupcakes should go next to each other (or they would get soggy!). How many ways could the course staff put  $n$  cupcakes into the tray (for arbitrary  $n$ )?

- (5) (a) Let  $c_n$  be the number of ways the course staff put  $n$  cupcakes into the tray. Write a recurrence equation for  $c_n$  and its initial conditions.

**Solution:**

- (10) (b) Find a generating function  $C(x) = \sum_{n=0}^{\infty} c_n x^n$  for  $c_n$ .

**Solution:**

- (5) (c) Using the above generating function  $C(x)$ , find a closed form for  $c_n$ .

**Solution:**

### 3. Pizza cleanup

Patrick and Andy were eating a pizza when they accidentally dropped the pizza on the floor, making a big mess! After getting the pizza off the floor, there were  $n$  pepperoni stains on the floor in a line, evenly spaced 1 foot apart. They have a mop that is 1.5 feet wide that they can use to clean up the mess, and they decide to play a game to see who has to pay for the next pizza.

They take turns moving the mop once across the floor, perpendicular to the line of the stains. Because of the width of the mop, each person can either clean up one stain, or clean up two stains that were originally next to each other. In every move, at least one stain needs to be cleaned up. The player to clean up the last stain wins. Andy gets to go first.

- (5) (a) Prove that if  $n > 0$ , then Andy will always win under optimal play.

**Solution:**

- (10) (b) Suppose  $n = 16$  and the stains are numbered 1 through 16. Suppose Andy starts by cleaning up spot 6, and then Patrick responds by cleaning up spot 2. Calculate the Nimber of the resulting position, and use it to determine which player will win, assuming both players play optimally from now on.

**Solution:**

#### 4. Two Player Games

- (15) (a) David and Taehoon are playing chess, when David decides to make it more interesting. He proposes that each turn, each person gets to make two moves instead of one.

Furthermore, David decides to raise the stakes. If Taehoon loses, he has to pay David one dollar. If David loses, David has to pay Taehoon two dollars. If they tie, Taehoon pays David 50 cents.

David plays as white (he gets to go first). Assuming both David and Taehoon want to win money, should Taehoon play?

<b>Solution:</b>
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- (20) (b) Victor and Venkat play the following game with apples and oranges:

We have apples and oranges lined up in a row. In each person's turn, he can remove a sequence of consecutive fruits from the rightmost side of the row.

For example, if we had  $AAOOAAAAOOOO$ , where  $A$  are apples and  $O$  are oranges, then for Victor's first move, he can remove either 1, 2, 3, 4 oranges from the right side.

If it were  $AAAOOOAAAOA$ , Victor would be forced to remove the rightmost apple.

Either player wins if he is the person to take the last fruit.

What are the  $P$  and  $N$  positions in this game?

<b>Solution:</b>
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