15-251 : Great Theoretical Ideas In Computer Science

Fall 2014

Assignment 1

Due: Thursday, Sep. 04, 2014 11:59 PM

Name: ____________________

Andrew ID: ____________________

Question:	1	2	3	4	5	Total
Points:	0	20	30	20	30	100
Score:						

1. ## Warmup

(a) Explain why you can go from ANY to ANY stack of pancakes in P_n steps..

(b) Use the breaking apart argument to give a lower bound on the number of flips required to sort the following stack of pancakes: 9,3,6,2,8,7,1,4,5. Here, 5 is assumed to be the pancake at the bottom and 9 is the one at the top.

(c) Prove by induction that for any positive integer n, the number $n^3 - n$ is a multiple of 3.

(d) Let's define n-factorial recursively as $n! = n \times (n-1)!$ and $0! = 1$. Show that

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$$

for all integers $n \geq 1$.

(e) Prove that for all $n \in \{1, 2, 3, \ldots\}$, $9^n - 8n - 1$ is divisible by 64.

(f) Alice has a two-dimensional $m \times n$ rectangular grid made of delicious chocolate. Each day, she takes a piece of chocolate, breaks it along some axis-aligned line into two integer sub-rectangles, and then eats all of the resulting pieces of size 1×1. How many days does it take her to eat all of her chocolate?

2. Pancakes with Problems

(10) (a) Suppose we have a special spatula which only lets us flip an even number of pancakes (from the top of the stack) each time. Show that the minimum number of flips needed using this spatula to sort any sortable stack of $2n$ pancakes is at least P_n and at most $3n - 2$ for all $n \geq 1$. (By "sortable" we mean that it is possible to sort the stack using the special spatula)

Solution:

(10) (b) Suppose now we are allowed to take any contiguous set of pancakes and flip them in place (they need not be on the top of the stack). Let Q_n be the maximum over stacks of size n of the minimum number of flips required to sort that stack, using this new flipping operation. Show that $n/2 \leq Q_n \leq n - 1$ for all $n \geq 2$.

Solution:

3. Bad Induction Proofs

For each of the proposed claims below, examine the proposed proof and point out the flaw in it. Do not just explain why the claim is wrong; rather you should explain how the argument violates the notion of a valid proof.

(10) (a) **Claim:** $\ln 2$ is a rational number. (Note that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. This is done by proving that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}$ is rational.

Proof by induction on n.

- *Base case:* $n = 1$: 1 is obviously rational.
- *Inductive hypothesis:* Suppose that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{k+1}}{k}$ is rational.
- *Inductive step:* We need to show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{k+2}}{k+1}$. is a rational number. Observe that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{k+2}}{k+1} = (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{k+1}}{k}) + \frac{(-1)^{k+2}}{k+1}$. By the induction hypothesis, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{k+1}}{k}$ is a rational number. Furthermore, $\frac{(-1)^{k+2}}{k+1}$ is a rational number, as it can be expressed in a fraction. Thus, summing two rational numbers will result in another rational number, and by induction we have proven that $\ln 2$ is rational.

Solution:

(10) (b) **Claim:** $\log_{15} n = \log_{251} n$ for all natural numbers n.

Proof (by strong induction)

The inductive hypothesis is "$\log_{15} n = \log_{251} n$".

- *Base case:* $\log_{15} 1 = 0 = \log_{251} 1$.
- *Induction Hypothesis:* $\log_{15} k = \log_{251} k$ for all natural numbers $k \leq n$
- *Inductive step:* We wish to show that the claim is true for $n + 1$. Write $n + 1$ as a product of two natural numbers p and q so that we have:

$$\log_{15}(n+1) = \log_{15}(pq) = \log_{15} p + \log_{15} q = \log_{251} p + \log_{251} q = \log_{251}(pq) = \log_{251}(n+1)$$

which is true by the inductive hypothesis.

Solution:

(10) (c) **Claim:**

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(n-1) \times n} = \frac{3}{2} - \frac{1}{n}$$

Proof by induction on n.

- *Base case:* For $n = 1$, $\frac{3}{2} - \frac{1}{n} = \frac{1}{1 \times 2}$.
- *Induction Hypothesis:* Assume the theorem is true for n.

- *Inductive step:*

$$\begin{aligned}
\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{(n-1)\times n} + \frac{1}{n\times(n+1)} &= \frac{3}{2} - \frac{1}{n} + \frac{1}{n\times(n+1)} \\
&= \frac{3}{2} - \frac{1}{n} + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{3}{2} - \frac{1}{n+1}
\end{aligned}$$

4. Features of Fibonacci

For these problems, use $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

(10) (a) Prove that $\forall n \geq 0, F_{2n+1} = F_{n+1}^2 + F_n^2$.

Solution:

(10) (b) Prove that $\forall n \geq 0$, $\sum_{i=0}^{n} \binom{n-i}{i} = F_{n+1}$.

Solution:

5. Induction Problems

(15) (a) Several straight lines and circles are drawn on a plane. Prove that one can paint the regions, created in result of intersection of the curves into black and white colors so that adjacent regions are of the different colors.

Solution:

(15) (b) You are manufacturing super cool locks that takes a combination sequence of n numbers from 0 to 9 to unlock (the numbers may be repeated). You decided to name the locks as Super Cool n-Locks. They're super cool, but there's a flaw in the design, so that the locks cannot have a combination where any TWO consecutive numbers are both even or both odd. How many unique Super Cool n-Locks can you make as a function of n? Prove the correctness of your claim by induction.

Solution: