Carnegie Mellon

School of Computer Science

Deep Reinforcement Learning and Control

Natural Policy Gradients

CMU 10-403

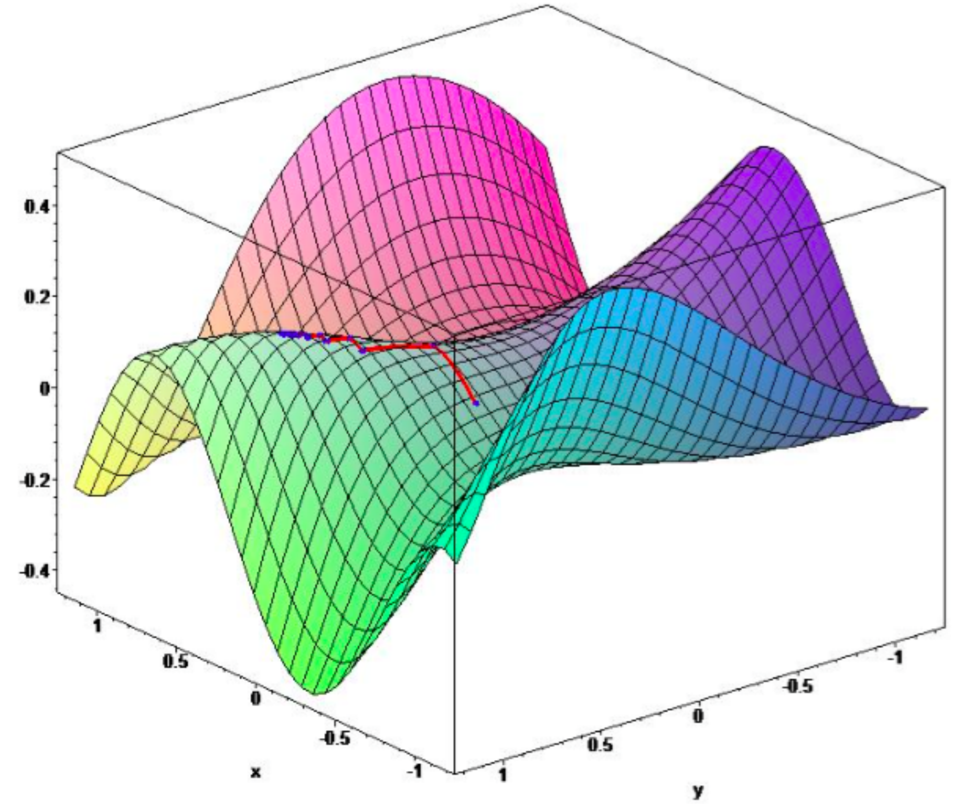
Katerina Fragkiadaki

Revision

Policy Gradient

- Let U(θ) be any policy **objective function**

- Policy gradient algorithms search for a local maximum in U(θ) by ascending the gradient of the policy, w.r.t. parameters θ



$$\theta_{new} = \theta_{old} + \Delta\theta$$

$$\Delta\theta = \alpha \nabla_\theta U(\theta)$$

Previous 3 lectures were about estimating/ computing this

α is a step-size parameter (learning rate)

is the policy gradient

$$\nabla_\theta U(\theta) = \begin{pmatrix} \frac{\partial U(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial U(\theta)}{\partial \theta_n} \end{pmatrix}$$

Policy gradient: the gradient of the policy objective w.r.t. the parameters of the policy

Computing the policy gradient

Likelihood ratio gradient estimator

$$\max_{\theta}. \quad U(\theta) = \mathbb{E}_{x \sim P_\theta(x)} f(x)$$

$$\max_{\theta}. \; U(\theta) = \mathbb{E}_{\tau \sim P_\theta(\tau)} \left[R(\tau) \right]$$

$$\nabla U(\theta) = \mathbb{E}_{x \sim P_\theta(x)} \nabla_\theta \log P_\theta(x) f(x)$$

$$\nabla U(\theta) = \mathbb{E}_{\tau \sim P_\theta(\tau)} \left[\nabla_\theta \log P_\theta(\tau) R(\tau) \right]$$

Derivatives of expectations

$$\begin{aligned}
\nabla_\theta \mathbb{E}_x f(x) &= \nabla_\theta \mathbb{E}_{x\sim P_\theta(x)}\left[f(x)\right] \\
&= \nabla_\theta \sum_x P_\theta(x) f(x) \\
&= \sum_x \nabla_\theta P_\theta(x) f(x) \\
&= \sum_x P_\theta(x) \frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} f(x) \\
&= \sum_x P_\theta(x)\, \nabla_\theta \log P_\theta(x) f(x) \\
&= \mathbb{E}_{x\sim P_\theta(x)}\left[\nabla_\theta \log P_\theta(x) f(x)\right] \\
&\approx \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log P_\theta(x^{(i)}) f(x^{(i)})
\end{aligned}$$

From the law of large numbers, I can obtain an unbiased estimator for the gradient by sampling!

Computing the policy gradient

Likelihood ratio gradient estimator

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Chain rule of derivatives

$$y = P_\theta(x)$$

$$\max_\theta . \; U(\theta) = f(P_\theta(x))$$

$$\nabla U(\theta) = \frac{df(P_\theta(x))}{d\theta} = \frac{df(y)}{dy} \frac{dy}{d\theta}$$

$$a = \pi_\theta(s)$$

$$\max_\theta . \quad U(\theta) = \mathbb{E} \sum_t Q^\pi(S_t, \pi_\theta(S_t))$$

$$\nabla U(\theta) = \frac{d \mathbb{E} \sum_t Q^\pi(S_t, \pi_\theta(S_t))}{d\theta} = \mathbb{E} \sum_t \frac{dQ^\pi(S_t, a)}{da} \frac{d\pi_\theta(S_t)}{d\theta}$$

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$$\max_{\theta, \phi} \quad U(\theta, \phi) = \mathbb{E} \sum_t Q_\phi(S_t, \pi_\theta(S_t))$$

$$\frac{\partial U(\theta, \phi)}{\partial \theta} = \frac{\partial \mathbb{E} \sum_t Q_\phi(S_t, \pi_\theta(S_t))}{\partial \theta} = \mathbb{E} \sum_t \frac{\partial Q_\phi(S_t, a)}{\partial a} \frac{d\pi_\theta(S_t)}{d\theta}$$

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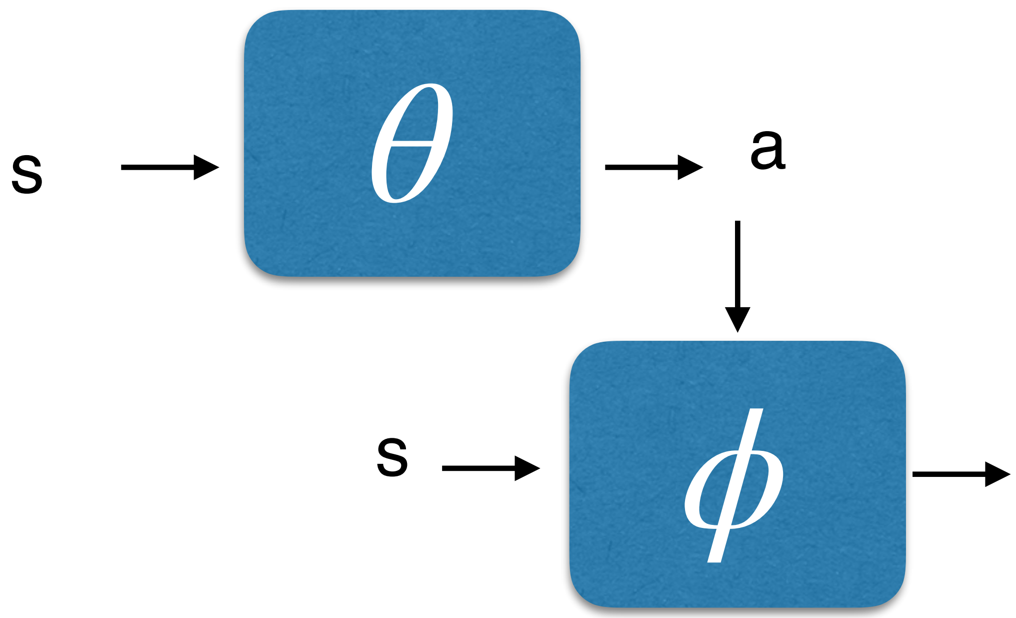
Deep Deterministic Policy Gradients

$$\frac{\partial U(\theta,\phi)}{\partial \theta} = \frac{\partial \mathbb{E}\sum_t Q_\phi(S_t, \pi_\theta(S_t))}{\partial \theta} = \mathbb{E}\sum_t \frac{\partial Q_\phi(S_t, a)}{\partial a}\frac{d\pi_\theta(S_t)}{d\theta}$$

$$\frac{\partial U(\theta,\phi)}{\partial \phi} = \frac{\partial \mathbb{E}\sum_t Q_\phi(S_t, \pi_\theta(S_t))}{\partial \phi} = \mathbb{E}\sum_t \frac{\partial Q_\phi(S_t, a)}{\partial \phi}$$



$$Q_\phi(s, \pi_\theta(a))$$

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Re-parametrization for Gaussian policies

$$\max_\theta . \quad U(\theta) = \mathbb{E}_{x \sim \mathcal{N}(\mu_\theta, \Sigma_\theta)} f(x)$$

$$\max_\theta . \quad U(\theta) = \mathbb{E}_{z \sim \mathcal{N}(0, I)} f(\mu_\theta + z * \sigma_\theta)$$

$$\max_\theta . \quad U(\theta) = \mathbb{E}_{A_t \sim \mathcal{N}(\mu_\theta(S_t), \sigma_\theta(S_t))} \sum_t Q^\pi(S_t, A_t)$$

$$\max_\theta . \quad U(\theta) = \mathbb{E}_{z \sim \mathcal{N}(0, I)} \sum_t Q^\pi \left(S_t, \mu_\theta(S_t) + z * \sigma_\theta(S_t) \right)$$

Re-parametrization for Gaussian

Instead of: $a \sim \mathcal{N}(\mu_\theta(s), \Sigma_\theta(s))$

We can write: $a = \mu_\theta(s) + z \odot \sigma_\theta(s)$ $\qquad z \sim \mathcal{N}(0, I)$

Why?

Because:

$$\mathbb{E}_z(\mu_\theta(s) + z\sigma_\theta(s)) = \mu_\theta(s)$$
$$\mathrm{Var}_z(\mu_\theta(s) + z\sigma_\theta(s)) = \sigma_\theta(s)^2$$

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$$\max_\theta . \quad U(\theta) = \mathbb{E}_{z \sim \mathcal{N}(0,I)} f(\mu_\theta + z * \sigma_\theta)$$

$$\max_{\theta,\phi} \quad U(\theta, \phi) = \mathbb{E}_{A_t \sim \mathcal{N}\left(\mu_\theta(S_t), \sigma_\theta(S_t)\right)} \sum_t Q_\phi(S_t, A_t)$$

$$\max_{\theta,\phi} \quad U(\theta, \phi) = \mathbb{E}_{z \sim \mathcal{N}(0,I)} \sum_t Q_\phi(S_t, \mu_\theta(S_t) + z\sigma_\theta(s))$$

$$\frac{\partial U(\theta,\phi)}{\partial \theta} = \frac{\partial \mathbb{E}_{z \sim \mathcal{N}(0,I)} \sum_t Q_\phi(S_t, \mu_\theta(S_t) + z\sigma_\theta(s))}{\partial \theta} = \mathbb{E}_{z \sim \mathcal{N}(0,I)} \sum_t \frac{\partial Q_\phi(S_t, a)}{\partial a} \frac{d(\mu_\theta(S_t) + z\sigma_\theta(s))}{d\theta}$$

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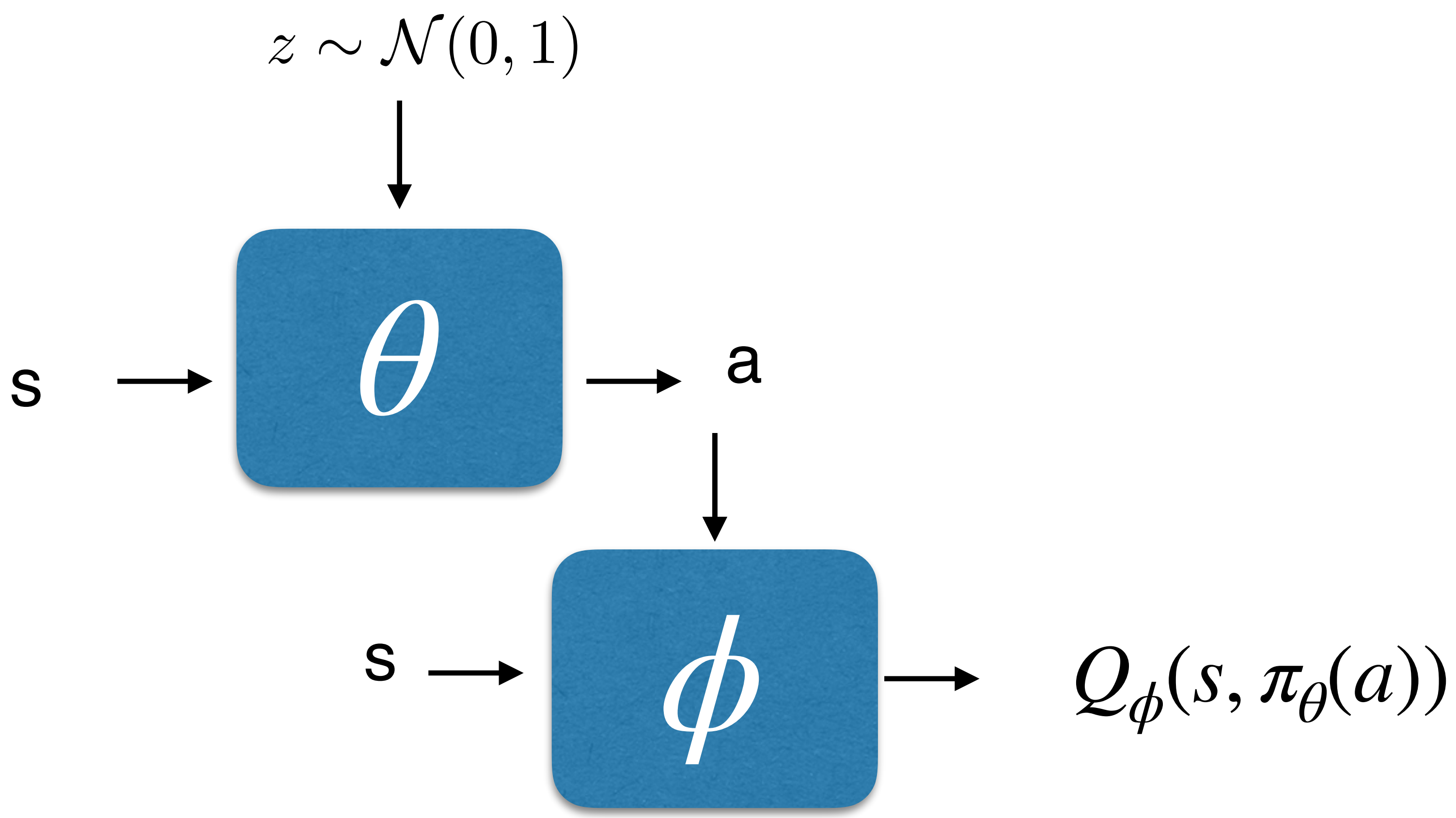
Stochastic Value Gradients



$$\frac{\partial U(\theta,\phi)}{\partial \theta} = \frac{\partial \mathbb{E}_{z\sim\mathcal{N}(0,I)} \sum_t Q_\phi(S_t, \mu_\theta(S_t) + z\sigma_\theta(s))}{\partial \theta} = \mathbb{E}_{z\sim\mathcal{N}(0,I)} \sum_t \frac{\partial Q_\phi(S_t, a)}{\partial a} \frac{d(\mu_\theta(S_t) + z\sigma_\theta(s))}{d\theta}$$

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Learning continuous control by stochastic value gradients, Hees et al.

Actor-critic

1. Sample trajectories $\{s_t^i, a_t^i\}_{i=0}^T$ by running the current policy $a \sim \pi_\theta(s)$
2. Fit value function $V_\phi^\pi(s)$ by MC or TD estimation (update ϕ)
3. Compute advantages $A^\pi(s_t^i, a_t^i) = R(s_t^i, a_t^i) + \gamma V_\phi^\pi(s_{t+1}^i) - V_\phi^\pi(s_t^i)$
4. $\nabla_\theta U(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^i \mid s_t^i) A^\pi(s_t^i, a_t^i)$
5. $\theta' = \theta + \alpha \nabla_\theta U(\theta)$

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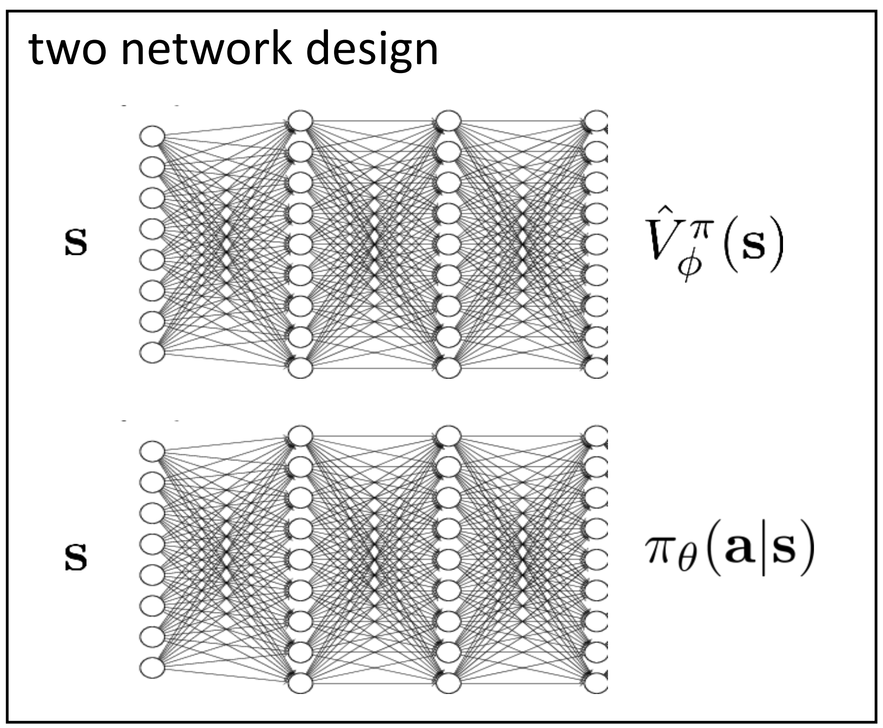

+ simple & stable

- no shared features between actor & critic

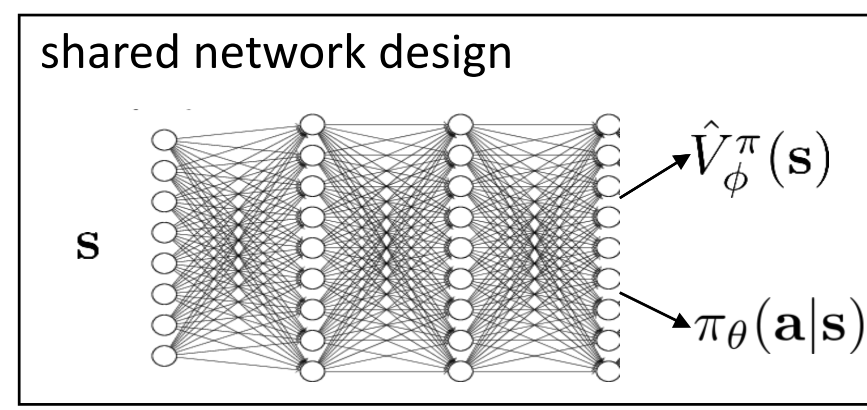

Figure from Sergey Levine

Actor-critic

1. Sample trajectories $\{s_t^{(i)}, a_t^{(i)}\}_{i=0}^{T}$ by running the current policy $a \sim \pi_\theta(s)$
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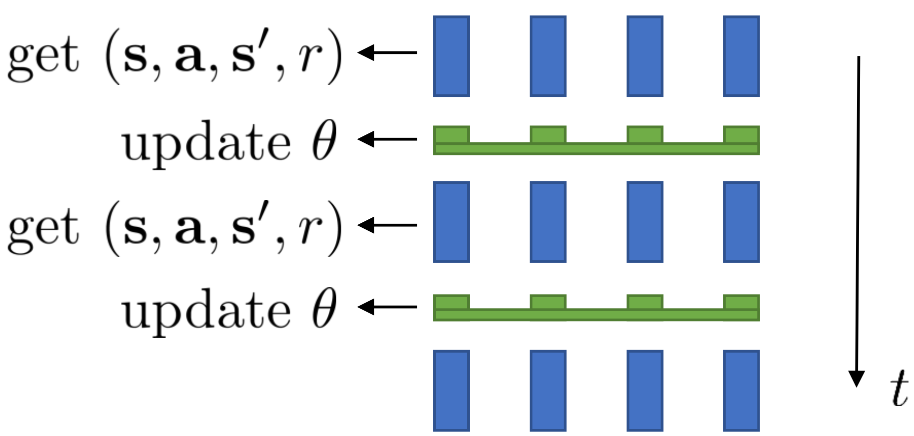
synchronized parallel actor-critic



asynchronous parallel actor-critic

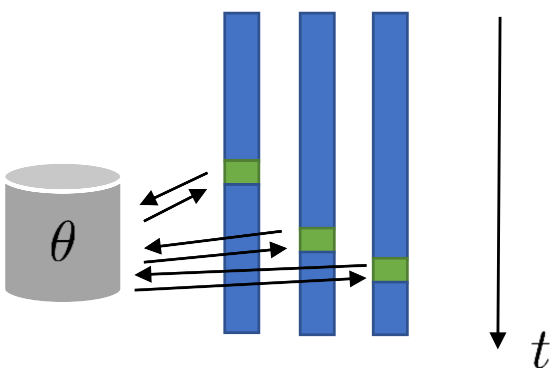


Figure from Sergey Levine

Critics are state dependent baselines

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)})(G_t^i - b)$$

+ no bias
- higher variance (because single-sample estimate)

$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^{(i)} \mid s_t^{(i)})(G_t^i - b(s_t^{(i)}))$$

+ no bias

+ lower variance (baseline is closer to rewards)

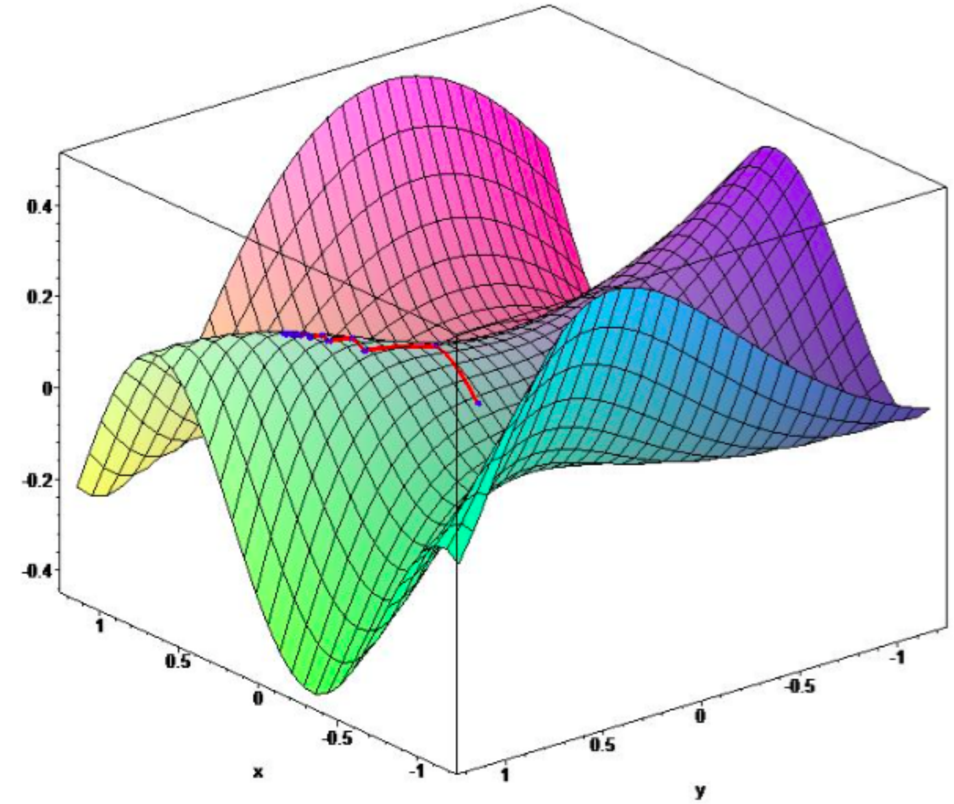
$$\hat{g} = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)}) \left(R(s_t^{(i)}, a_t^{(i)}) + \gamma V^\pi(s_{t+1}^{(i)}) - V^\pi(s_t^{(i)}) \right)$$

+ lower variance (due to critic)
- not unbiased (if the critic is not perfect)

Figure from Sergey Levine

Policy Gradient

- Let U(θ) be any policy **objective function**
- Policy gradient algorithms search for a local maximum in U(θ) by ascending the gradient of the policy, w.r.t. parameters θ



$$\theta_{new} = \theta_{old} + \Delta\theta$$

$$\Delta\theta = \alpha \nabla_\theta U(\theta)$$

This lecture is all about the stepsize

α is a step-size parameter (learning rate)

is the policy gradient

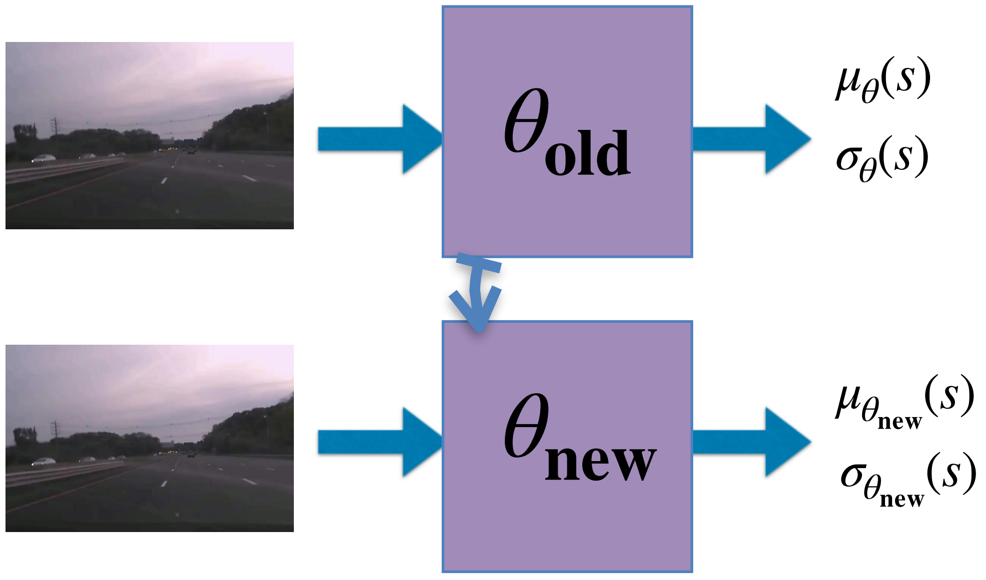
$$\nabla_\theta U(\theta) = \begin{pmatrix} \frac{\partial U(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial U(\theta)}{\partial \theta_n} \end{pmatrix}$$

Policy gradient: the gradient of the policy objective w.r.t. the parameters of the policy

Policy Gradients

1. Collect trajectories for policy π_θ
2. Estimate advantages A
3. Compute policy gradient $\hat{g}$
4. Update policy parameters $\theta_{new} = \theta + \epsilon \cdot \hat{g}$
5. GOTO 1

This lecture is all about the stepsize



What is the underlying objective function?

Policy gradients:

$$\hat{g} \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)}) A(s_t^{(i)}, a_t^{(i)}), \quad \tau_i \sim \pi_\theta$$

This result from differentiating the following objective function:

$$U^{PG}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \log \pi_\theta(\alpha_t^{(i)} \mid s_t^{(i)}) A(s_t^{(i)}, a_t^{(i)}) \quad \tau_i \sim \pi_\theta$$

Compare this to supervised learning using expert actions $\tilde{a} \sim \pi^*$ and a maximum likelihood objective:

$$U^{SL}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \log \pi_\theta(\tilde{\alpha}_t^{(i)} \mid s_t^{(i)}), \quad \tau_i \sim \pi^*$$ (+regularization)

This maximizes the probability of expert actions in the training set.

We want to optimize both objectives using gradient descent

$$\theta' = \theta + \alpha \nabla_\theta U(\theta)$$

Choosing stepsize a is more critical for RL than for SL.

Why?

Because we cannot optimize it too far, our advantage estimates come from $\pi_{\theta_{old}}$

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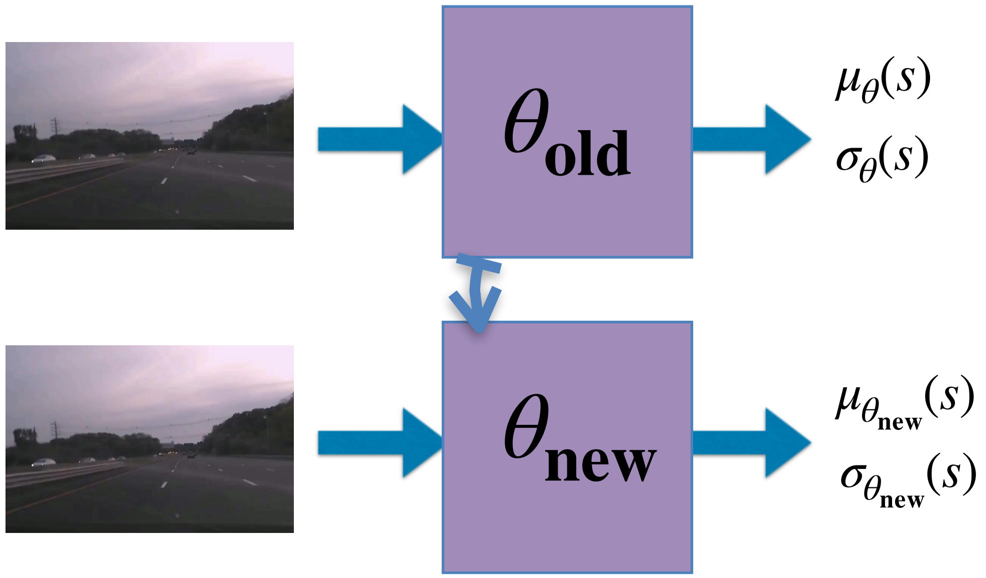
Two problems:
1. Hard to choose stepwise ϵ
2. Sample inefficient: we cannot use data collected with policies of previous iterations



Hard to choose stepsizes

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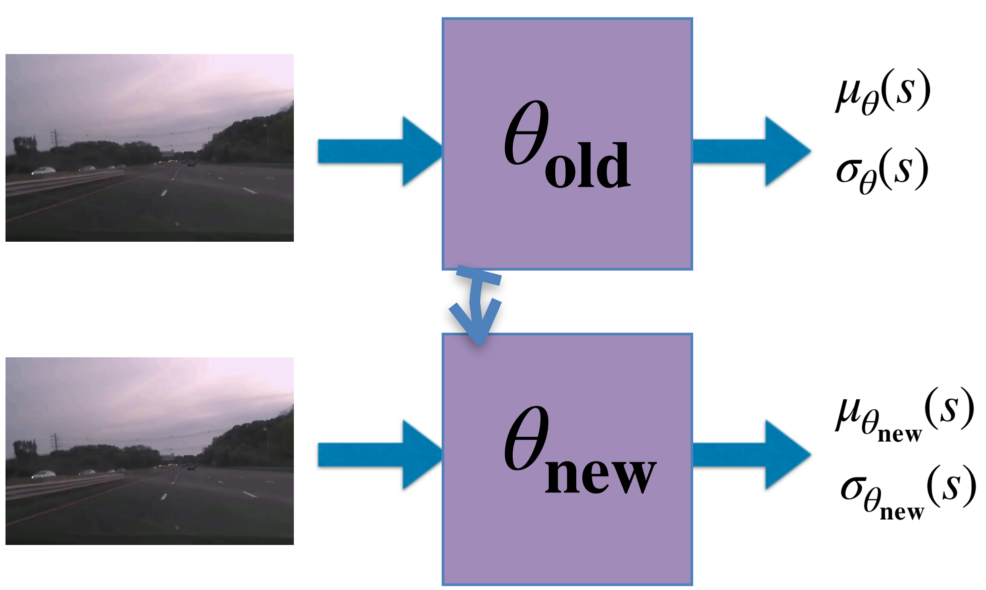
- Step too big
  Bad policy->data collected under bad policy-> we cannot recover
  (in Supervised Learning, data does not depend on neural network weights)
- Step too small
  Not efficient use of experience
  (in Supervised Learning, data can be trivially re-used)

Gradient descent in parameter space does not take into account the resulting distance in the (output) policy space between $\pi_{\theta_{\text{old}}}(s)$ and $\pi_{\theta_{\text{new}}}(s)$



Hard to choose stepsizes

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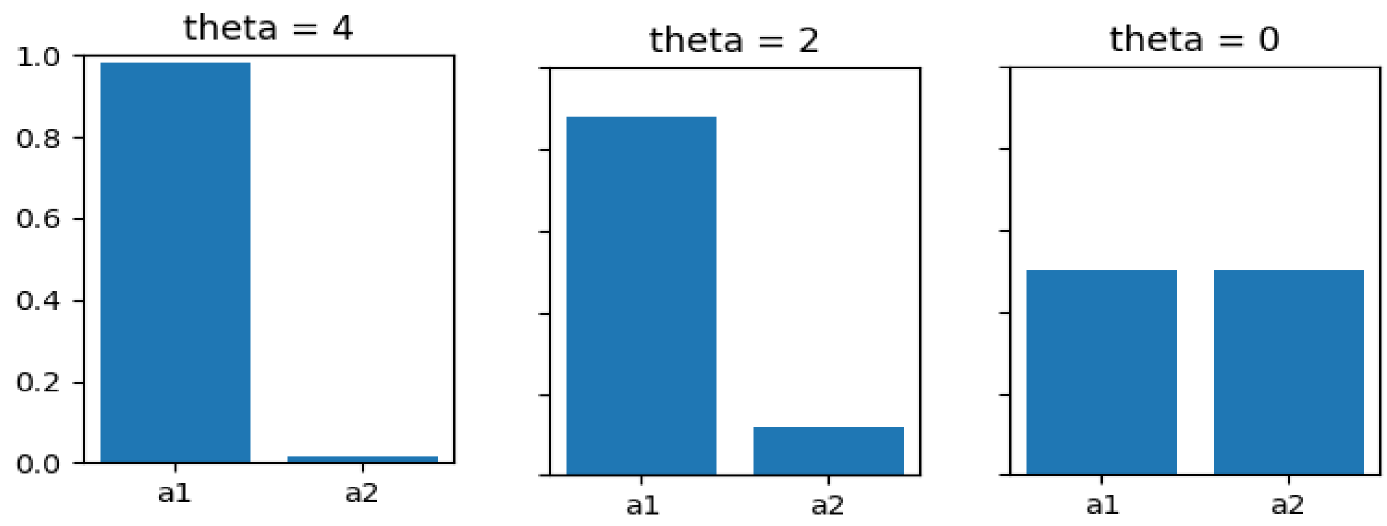
Consider a family of policies with parametrization:

$$\pi_\theta(a) = \begin{cases} \sigma(\theta) & a = 1 \\ 1 - \sigma(\theta) & a = 2 \end{cases}$$



The same parameter step $\Delta\theta = -2$ changes the policy distribution more or less dramatically depending on where in the parameter space we are.

Notation

We will use the following to denote values of parameters and corresponding policies before and after an update:

$$\theta_{old} \to \theta_{new}$$

$$\pi_{old} \to \pi_{new}$$

$$\theta \to \theta'$$

$$\pi \to \pi'$$

Gradient Descent in Parameter Space

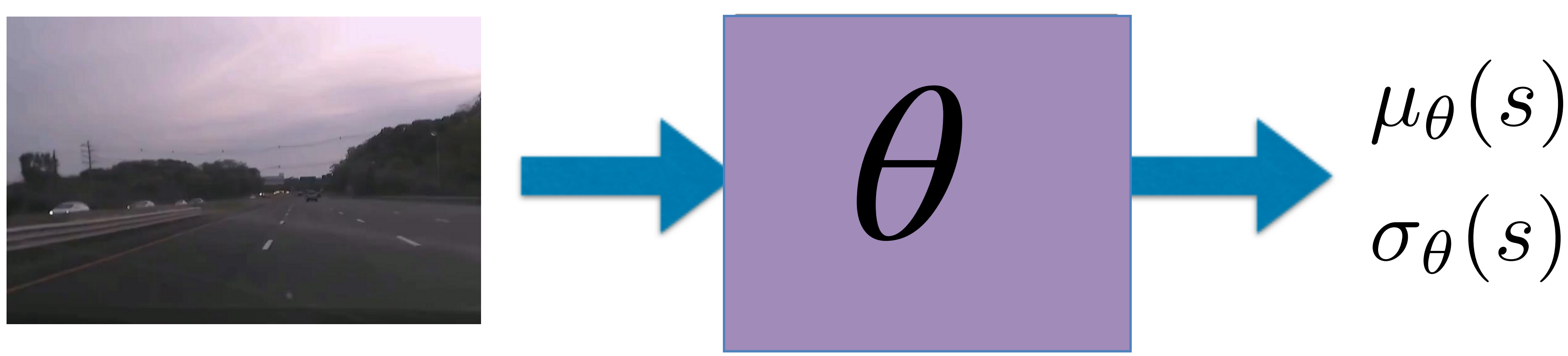
The stepwise in gradient descent results from solving the following optimization problem, e.g., using line search:

$$d^* = \arg\max_{\|d\| \leq \epsilon} J(\theta + d)$$

SGD: $\theta_{new} = \theta_{old} + d^*$

Euclidean distance in parameter space

It is hard to predict the result on the parameterized distribution.. hard to pick the threshold epsilon



Gradient Descent in Distribution Space

The stepwise in gradient descent results from solving the following optimization problem, e.g., using line search:

$$d* = \arg \max_{\|d\| \leq \epsilon} J(\theta + d)$$

SGD: $\theta_{new} = \theta_{old} + d*$

Euclidean distance in parameter space

It is hard to predict the result on the parameterized distribution.. hard to pick the threshold epsilon

Natural gradient descent: the stepwise in parameter space is determined by considering the KL divergence in the distributions before and after the update:

$$\boxed{d* = \arg \max_{d,\ s.t.\ \mathrm{KL}(\pi_\theta \| \pi_{\theta+d}) \leq \epsilon} J(\theta + d)}$$

KL divergence in distribution space

Easier to pick the distance threshold!!!

$$D_{\mathrm{KL}}(P\|Q) = \sum_i P(i) \log\left(\frac{P(i)}{Q(i)}\right)$$

$$D_{\mathrm{KL}}(P\|Q) = \int_{-\infty}^{\infty} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

Solving the KL Constrained Problem

Unconstrained penalized objective:

$$d^* = \arg\max_d U(\theta + d) - \lambda(\mathrm{D_{KL}}\left[\pi_\theta \| \pi_{\theta+d}\right] - \epsilon)$$

First order Taylor expansion for the loss and second order for the KL:

$$\approx \arg\max_d U(\theta_{old}) + \nabla_\theta U(\theta)\,|_{\theta=\theta_{old}} \cdot d - \frac{1}{2}\lambda(d^\top \nabla_\theta^2 \mathrm{D_{KL}}\left[\pi_{\theta_{old}} \| \pi_\theta\right]|_{\theta=\theta_{old}} d) + \lambda\epsilon$$

Taylor expansion of KL

$$\mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta}) \approx \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta_{old}}) + d^{\top} \nabla_{\theta} \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta}) |_{\theta=\theta_{old}} + \frac{1}{2} d^{\top} \nabla_{\theta}^{2} \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta}) |_{\theta=\theta_{old}} d$$

$$\mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta}) = \mathbb{E}_{x \sim p_{\theta_{old}}} \log \left(\frac{P_{\theta_{old}}(x)}{P_{\theta}(x)} \right)$$

Taylor expansion of KL

$$D_{KL}(p_{\theta_{old}}|p_\theta) \approx D_{KL}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2}d^\top \nabla_\theta^2 D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$D_{KL}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)$$

Taylor expansion of KL

$$D_{KL}(p_{\theta_{old}}|p_\theta) \approx D_{KL}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \boxed{\nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}}} + \frac{1}{2} d^\top \nabla^2_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\boxed{\nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}}} = -\nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_\theta(x)|_{\theta=\theta_{old}} + \nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_{\theta_{old}}(x)|_{\theta=\theta_{old}}$$

$$\boxed{D_{KL}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)}$$

Taylor expansion of KL

$$\mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta) \approx \mathrm{D_{KL}}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla_\theta^2 \mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}\nabla_\theta \mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_\theta(x)|_{\theta=\theta_{old}} + \nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_{\theta_{old}}(x)|_{\theta=\theta_{old}} \\ &= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x)|_{\theta=\theta_{old}}\end{aligned}$$

$$\mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)$$

Taylor expansion of KL

$$D_{KL}(p_{\theta_{old}}|p_\theta) \approx D_{KL}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla_\theta^2 D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_\theta(x)|_{\theta=\theta_{old}} + \nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_{\theta_{old}}(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}}
\end{aligned}$$

$$D_{KL}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)$$

Taylor expansion of KL

$$\mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta) \approx \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla_\theta^2 \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_\theta(x)|_{\theta=\theta_{old}} + \nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_{\theta_{old}}(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}} \\
&= \int_x P_{\theta_{old}}(x) \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}}
\end{aligned}$$

$$\mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)$$

Taylor expansion of KL

$$\mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta) \approx \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla_\theta^2 \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_\theta(x)|_{\theta=\theta_{old}} + \nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_{\theta_{old}}(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}} \\
&= \int_x P_{\theta_{old}}(x) \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}} \\
&= \int_x \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}}
\end{aligned}$$

$$\mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)$$

Taylor expansion of KL

$$D_{KL}(p_{\theta_{old}}|p_\theta) \approx D_{KL}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla_\theta^2 D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_\theta(x)|_{\theta=\theta_{old}} + \nabla_\theta \mathbb{E}_{x\sim p_{\theta_{old}}} \log P_{\theta_{old}}(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}} \\
&= \int_x P_{\theta_{old}}(x) \frac{1}{P_{\theta_{old}}(x)} \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}} \\
&= \int_x \nabla_\theta P_\theta(x)|_{\theta=\theta_{old}} \\
&= \nabla_\theta \int_x P_\theta(x)|_{\theta=\theta_{old}} \,. \\
&= 0
\end{aligned}$$

$$D_{KL}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log\left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)}\right)$$

Taylor expansion of KL

$$D_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta) \approx D_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{KL}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla_\theta^2 D_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} d$$

$$\nabla_\theta^2 D_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} = -\mathbb{E}_{x \sim p_{\theta_{old}}} \nabla_\theta^2 \log P_\theta(x)|_{\theta=\theta_{old}}$$

$$D_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta) = \mathbb{E}_{x \sim p_{\theta_{old}}} \log \left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)} \right)$$

Taylor expansion of KL

$$\mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta) \approx \mathrm{D_{KL}}(p_{\theta_{old}} | p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{KL}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla^2_\theta \mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}\nabla^2_\theta \mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} &= -\mathbb{E}_{x \sim p_{\theta_{old}}} \nabla^2_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\ &= -\mathbb{E}_{x \sim p_{\theta_{old}}} \nabla_\theta \left(\frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} \right)|_{\theta=\theta_{old}}\end{aligned}$$

$$\mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta) = \mathbb{E}_{x \sim p_{\theta_{old}}} \log \left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)} \right)$$

Taylor expansion of KL

$$\mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta) \approx \mathrm{D_{KL}}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla^2_\theta \mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla^2_\theta \mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla^2_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \left(\frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} \right)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \left(\frac{\nabla^2_\theta P_\theta(x) P_\theta(x) - \nabla_\theta P_\theta(x) \nabla_\theta P_\theta(x)^\top}{P_\theta(x)^2} \right)|_{\theta=\theta_{old}}
\end{aligned}$$

$$\mathrm{D_{KL}}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log \left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)} \right)$$

Taylor expansion of KL

$$\mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta) \approx \mathrm{D_{KL}}(p_{\theta_{old}} | p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{KL}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla^2_\theta \mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla^2_\theta \mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta)|_{\theta=\theta_{old}} &= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla^2_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \left(\frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} \right)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \left(\frac{\nabla^2_\theta P_\theta(x) P_\theta(x) - \nabla_\theta P_\theta(x) \nabla_\theta P_\theta(x)^\top}{P_\theta(x)^2} \right)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \frac{\nabla^2_\theta P_\theta(x)|_{\theta=\theta_{old}}}{P_{\theta_{old}}(x)} + \mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x) \nabla_\theta \log P_\theta(x)^\top |_{\theta=\theta_{old}}
\end{aligned}$$

$$\mathrm{D_{KL}}(p_{\theta_{old}} | p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log \left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)} \right)$$

Taylor expansion of KL

$$D_{KL}(p_{\theta_{old}}|p_\theta) \approx D_{KL}(p_{\theta_{old}}|p_{\theta_{old}}) + d^\top \nabla_\theta KL(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} + \frac{1}{2}d^\top \nabla^2_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} d$$

$$\begin{aligned}
\nabla^2_\theta D_{KL}(p_{\theta_{old}}|p_\theta)|_{\theta=\theta_{old}} &= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla^2_\theta \log P_\theta(x)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \left(\frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} \right)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \left(\frac{\nabla^2_\theta P_\theta(x) P_\theta(x) - \nabla_\theta P_\theta(x) \nabla_\theta P_\theta(x)^\top}{P_\theta(x)^2} \right)|_{\theta=\theta_{old}} \\
&= -\mathbb{E}_{x\sim p_{\theta_{old}}} \frac{\nabla^2_\theta P_\theta(x)|_{\theta=\theta_{old}}}{P_{\theta_{old}}(x)} + \mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x) \nabla_\theta \log P_\theta(x)^\top|_{\theta=\theta_{old}} \\
&= \mathbb{E}_{x\sim p_{\theta_{old}}} \nabla_\theta \log P_\theta(x) \nabla_\theta \log P_\theta(x)^\top|_{\theta=\theta_{old}}
\end{aligned}$$

$$D_{KL}(p_{\theta_{old}}|p_\theta) = \mathbb{E}_{x\sim p_{\theta_{old}}} \log \left(\frac{P_{\theta_{old}}(x)}{P_\theta(x)} \right)$$

Fisher Information Matrix

Exactly equivalent to the Hessian of KL divergence!

$$\mathbf{F}(\theta) = \mathbb{E}_\theta \left[\nabla_\theta \log p_\theta(x) \, \nabla_\theta \log p_\theta(x)^\top \right]$$

$$\mathbf{F}(\theta_{old}) = \nabla^2_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta) \,|_{\theta=\theta_{old}}$$

$$\begin{aligned} \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta) &\approx \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_{\theta_{old}}) + d^\top \nabla_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta) \,|_{\theta=\theta_{old}} + \frac{1}{2} d^\top \nabla^2_\theta \mathrm{D}_{\mathrm{KL}}(p_{\theta_{old}} | p_\theta) \,|_{\theta=\theta_{old}} d \\ &= \frac{1}{2} d^\top \mathbf{F}(\theta_{old}) d \\ &= \frac{1}{2} (\theta - \theta_{old})^\top \mathbf{F}(\theta_{old})(\theta - \theta_{old}) \end{aligned}$$

Since KL divergence is roughly analogous to a distance measure between distributions, Fisher information serves as a local distance metric between distributions: how much you change the distribution if you move the parameters a little bit in a given direction.

Solving the KL Constrained Problem

Unconstrained penalized objective:

$$d^* = \arg\max_d U(\theta + d) - \lambda(\mathrm{D_{KL}}\left[\pi_\theta \| \pi_{\theta+d}\right] - \epsilon)$$

First order Taylor expansion for the loss and second order for the KL:

$$\approx \arg\max_d U(\theta_{old}) + \nabla_\theta U(\theta)\,|_{\theta=\theta_{old}} \cdot d - \frac{1}{2}\lambda(d^\top \nabla_\theta^2 \mathrm{D_{KL}}\left[\pi_{\theta_{old}} \| \pi_\theta\right]|_{\theta=\theta_{old}} d) + \lambda\epsilon$$

Substitute for the information matrix:

$$= \arg\max_d \nabla_\theta U(\theta)\,|_{\theta=\theta_{old}} \cdot d - \frac{1}{2}\lambda(d^\top \mathbf{F}(\theta_{old})d)$$

$$= \arg\min_d - \nabla_\theta U(\theta)\,|_{\theta=\theta_{old}} \cdot d + \frac{1}{2}\lambda(d^\top \mathbf{F}(\theta_{old})d)$$

Natural Gradient Descent

Setting the gradient to zero:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial d}\left(-\nabla_\theta U(\theta)|_{\theta=\theta_{old}} \cdot d + \frac{1}{2}\lambda(d^\top \mathbf{F}(\theta_{old})d)\right) \\
&= -\nabla_\theta U(\theta)|_{\theta=\theta_{old}} + \frac{1}{2}\lambda(\mathbf{F}(\theta_{old}))d
\end{aligned}
$$

$$
d = \frac{2}{\lambda}\mathbf{F}^{-1}(\theta_{old})\nabla_\theta U(\theta)|_{\theta=\theta_{old}}
$$

The natural gradient:

$$
\boxed{\tilde{\nabla} J(\theta) = \mathbf{F}^{-1}(\theta_{old})\nabla_\theta J(\theta)}
$$

$$
\theta_{new} = \theta_{old} + \alpha \cdot \mathbf{F}^{-1}(\theta_{old})\hat{g}
$$

$$
\mathrm{D}_{\mathrm{KL}}(\pi_{\theta_{old}} | \pi_\theta) \approx \frac{1}{2}(\theta - \theta_{old})^\top \mathbf{F}(\theta_{old})(\theta - \theta_{old})
$$

$$
\frac{1}{2}(\alpha g_N)^\top \mathbf{F}(\alpha g_N) = \epsilon
$$

$$
\alpha = \sqrt{\frac{2\epsilon}{(g_N^\top \mathbf{F} g_N)}}
$$

Natural Gradient Descent

Algorithm 1 Natural Policy Gradient

Input: initial policy parameters θ_0

for $k = 0, 1, 2, \ldots$ **do**

Collect set of trajectories $\mathcal{D}_k$ on policy $\pi_k = \pi(\theta_k)$

Estimate advantages $\hat{A}_t^{\pi_k}$ using any advantage estimation algorithm

Form sample estimates for

- policy gradient $\hat{g}_k$ (using advantage estimates)
- and KL-divergence Hessian / Fisher Information Matrix $\hat{H}_k$

Compute Natural Policy Gradient update:

$$\theta_{k+1} = \theta_k + \sqrt{\frac{2\,\epsilon}{\hat{g}_k^T \hat{H}_k^{-1} \hat{g}_k}} \hat{H}_k^{-1} \hat{g}_k$$

end for

Both use samples from the current policy $\pi_k = \pi(\theta_k)$

Natural Gradient Descent

Algorithm 1 Natural Policy Gradient

Input: initial policy parameters θ_0
for $k = 0, 1, 2, \ldots$ **do**
Collect set of trajectories $\mathcal{D}_k$ on policy $\pi_k = \pi(\theta_k)$
Estimate advantages $\hat{A}_t^{\pi_k}$ using any advantage estimation algorithm
Form sample estimates for

- policy gradient $\hat{g}_k$ (using advantage estimates)
- and KL-divergence Hessian / Fisher Information Matrix $\hat{H}_k$

Compute Natural Policy Gradient update:

$$\theta_{k+1} = \theta_k + \sqrt{\frac{2\epsilon}{\hat{g}_k^T \hat{H}_k^{-1} \hat{g}_k}} \hat{H}_k^{-1} \hat{g}_k$$

end for

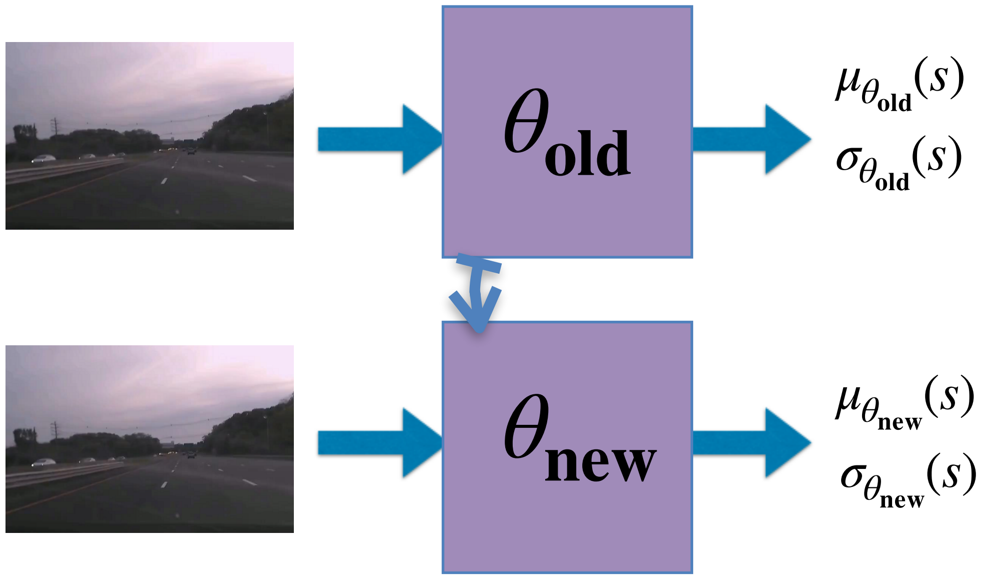
very expensive to compute for a large number of parameters!

Policy Gradients

Monte Carlo Policy Gradients (REINFORCE), gradient direction: $\hat{g} = \hat{\mathbb{E}}_t \left[\nabla_\theta \log \pi_\theta(a_t \mid s_t) \hat{A}_t \right]$

Actor-Critic Policy Gradient: $\hat{g} = \hat{\mathbb{E}}_t \left[\nabla_\theta \log \pi_\theta(a_t \mid s_t) A_{\mathbf{w}}(s_t) \right]$

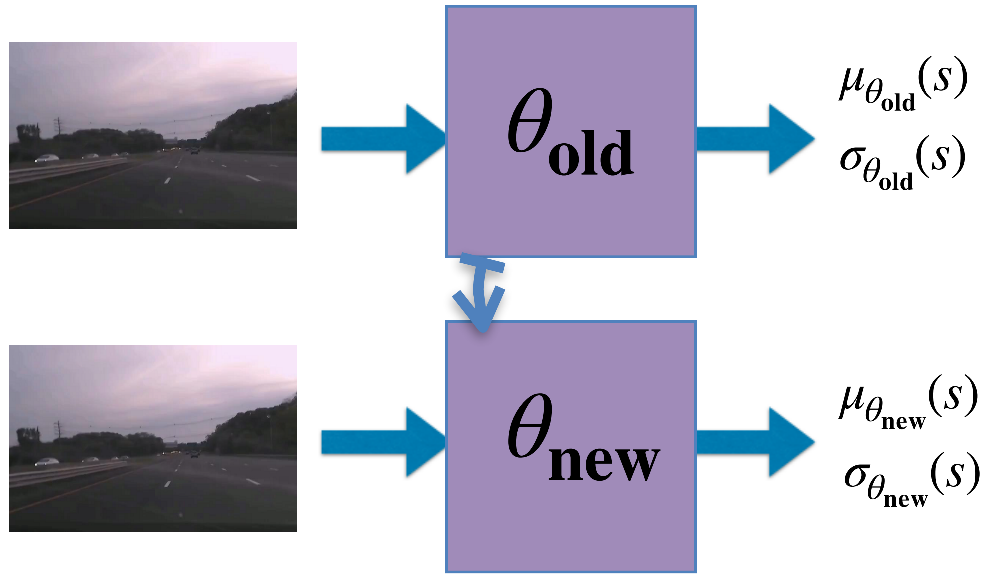
1. Collect trajectories for policy $\pi_{\theta_{old}}$
2. Estimate advantages A
3. Compute policy gradient $\hat{g}$
4. Update policy parameters $\theta_{new} = \theta_{old} + \epsilon \cdot \hat{g}$
5. GOTO 1



Policy Gradients

1. Collect trajectories for policy $\pi_{\theta_{old}}$
2. Estimate advantages A
3. Compute policy gradient $\hat{g}$
4. Update policy parameters $\theta_{new} = \theta_{old} + \epsilon \cdot \hat{g}$
5. GOTO 1

- On policy learning can be extremely inefficient
- The policy changes only a little bit with each gradient step
- I want to be able to use earlier data..how to do that?



Off-policy learning with Importance Sampling

$$
\begin{aligned}
U(\theta) &= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}\left[R(\tau)\right] \\
&= \sum_{\tau} \pi_\theta(\tau) R(\tau) \\
&= \sum_{\tau} \pi_{\theta_{old}}(\tau) \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau) \\
&= \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)
\end{aligned}
$$

$$
\nabla_\theta U(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} R(\tau)
$$

$$
\nabla_\theta U(\theta)\,|_{\theta=\theta_{old}} = \mathbb{E}_{\tau \sim \pi_{\theta_{old}}} \nabla_\theta \log \pi_\theta(\tau)\,|_{\theta=\theta_{old}} R(\tau)
$$

<-Gradient evaluated at theta_old is unchanged

Off policy learning with Importance Sampling

$$U(\theta) = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}\left[R(\tau)\right]$$
$$= \sum_\tau \pi_\theta(\tau)R(\tau)$$
$$= \sum_\tau \pi_{\theta_{old}}(\tau)\frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)}R(\tau)$$
$$= \sum_{\tau\sim\pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)}R(\tau)$$
$$= \mathbb{E}_{\tau\sim\pi_{\theta_{old}}} \frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)}R(\tau)$$

$$\frac{\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)} = \prod_{i=1}^{T}\frac{\pi_\theta(a_t|s_t)}{\pi_{\theta_{old}}(a_t|s_t)}$$

$$\Rightarrow \quad U(\theta) = \mathbb{E}_{\tau\sim\pi_{\theta_{old}}}\sum_{t=1}^{T}\prod_{t'=1}^{t}\frac{\pi_\theta(a'_t|s'_t)}{\pi_{\theta_{old}}(a'_t|s'_t)}\hat{A}_t$$

$$\nabla_\theta U(\theta) = \mathbb{E}_{\tau\sim\pi_{\theta_{old}}}\frac{\nabla_\theta\pi_\theta(\tau)}{\pi_{\theta_{old}}(\tau)}R(\tau)$$

$$\nabla_\theta U(\theta)|_{\theta=\theta_{old}} = \mathbb{E}_{\tau\sim\pi_{\theta_{old}}}\nabla_\theta \log \pi_\theta(\tau)|_{\theta=\theta_{old}}R(\tau)$$

Trust region Policy Optimization

Define the constrained objective:

$$\begin{aligned} \underset{\theta}{\text{maximize}} \quad & \hat{\mathbb{E}}_t \left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{\text{old}}}(a_t \mid s_t)} \hat{A}_t \right] \\ \text{subject to} \quad & \hat{\mathbb{E}}_t[\text{KL}[\pi_{\theta_{\text{old}}}(\cdot \mid s_t), \pi_\theta(\cdot \mid s_t)]] \leq \delta. \end{aligned}$$

- Also worth considering using a penalty instead of a constraint

$$\underset{\theta}{\text{maximize}} \quad \hat{\mathbb{E}}_t \left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{\text{old}}}(a_t \mid s_t)} \hat{A}_t \right] - \beta \hat{\mathbb{E}}_t[\text{KL}[\pi_{\theta_{\text{old}}}(\cdot \mid s_t), \pi_\theta(\cdot \mid s_t)]]$$

Again the KL penalized problem!

J. Schulman, S. Levine, P. Moritz, M. I. Jordan, and P. Abbeel. "Trust Region Policy Optimization".

Trust Region Policy Optimization

Police gradients with monotonic guarantees!

- Police gradients: have a function approximation for the policy $\pi_\theta(u|x)$ and optimize use SGD. SGD is sufficient to learn great object object detectors for example. What is different in RL?
- Non-stationarity in RL: *Each time the policy changes the state visitation distribution changes.* And this can cause the policy to diverge!
- Contribution: theoretical and practical method of how big of a step our gradient can take.

Trust Region Policy Optimization, Schulman et al. 2015

Trust region Policy Optimization

- $\text{maximize}_\theta \; L_{\pi_{\theta_{\text{old}}}}(\pi_\theta) - \beta \cdot \overline{\text{KL}}_{\pi_{\theta_{\text{old}}}}(\pi_\theta)$
- Make linear approximation to $L_{\pi_{\theta_{\text{old}}}}$ and quadratic approximation to KL term:

$$\underset{\theta}{\text{maximize}} \quad g \cdot (\theta - \theta_{\text{old}}) - \frac{\beta}{2}(\theta - \theta_{\text{old}})^T F(\theta - \theta_{\text{old}})$$

$$\text{where} \quad g = \frac{\partial}{\partial \theta} L_{\pi_{\theta_{\text{old}}}}(\pi_\theta)\big|_{\theta=\theta_{\text{old}}}, \quad F = \frac{\partial^2}{\partial^2 \theta} \overline{\text{KL}}_{\pi_{\theta_{\text{old}}}}(\pi_\theta)\big|_{\theta=\theta_{\text{old}}}$$

Exactly what we saw with natural policy gradient!
One important detail!

Trust region Policy Optimization

Due to the quadratic approximation, the KL constraint may be violated! What if we just do a line search to find the best stepsize, making sure:

- I am improving my objective J(\theta)
- The KL constraint is not violated!

$$\begin{aligned} \underset{\theta}{\text{maximize}} \quad & \hat{\mathbb{E}}_t\left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{\text{old}}}(a_t \mid s_t)}\hat{A}_t\right] \\ \text{subject to} \quad & \hat{\mathbb{E}}_t[\text{KL}[\pi_{\theta_{\text{old}}}(\cdot \mid s_t), \pi_\theta(\cdot \mid s_t)]] \leq \delta. \end{aligned}$$

Algorithm 2 Line Search for TRPO

Compute proposed policy step $\Delta_k = \sqrt{\frac{2\delta}{\hat{g}_k^T \hat{H}_k^{-1} \hat{g}_k}} \hat{H}_k^{-1} \hat{g}_k$

for $j = 0, 1, 2, ..., L$ **do**

  Compute proposed update $\theta = \theta_k + \alpha^j \Delta_k$

  if $\mathcal{L}_{\theta_k}(\theta) \geq 0$ and $\bar{D}_{KL}(\theta || \theta_k) \leq \delta$ **then**

    accept the update and set $\theta_{k+1} = \theta_k + \alpha^j \Delta_k$

    break

  end if

end for

Trust region Policy Optimization

TRPO= NPG +Linesearch

Algorithm 3 Trust Region Policy Optimization

Input: initial policy parameters θ_0

for $k = 0, 1, 2, \ldots$ **do**

Collect set of trajectories $\mathcal{D}_k$ on policy $\pi_k = \pi(\theta_k)$

Estimate advantages $\hat{A}_t^{\pi_k}$ using any advantage estimation algorithm

Form sample estimates for

- policy gradient $\hat{g}_k$ (using advantage estimates)
- and KL-divergence Hessian-vector product function $f(v) = \hat{H}_k v$

Use CG with n_{cg} iterations to obtain $x_k \approx \hat{H}_k^{-1} \hat{g}_k$

Estimate proposed step $\Delta_k \approx \sqrt{\frac{2\delta}{x_k^T \hat{H}_k x_k}} x_k$

Perform backtracking line search with exponential decay to obtain final update

$$\theta_{k+1} = \theta_k + \alpha^j \Delta_k$$

end for

Trust region Policy Optimization

TRPO= NPG +Linesearch+monotonic improvement theorem!

Algorithm 3 Trust Region Policy Optimization

Input: initial policy parameters θ_0

for $k = 0, 1, 2, \ldots$ **do**

Collect set of trajectories $\mathcal{D}_k$ on policy $\pi_k = \pi(\theta_k)$

Estimate advantages $\hat{A}_t^{\pi_k}$ using any advantage estimation algorithm

Form sample estimates for

- policy gradient $\hat{g}_k$ (using advantage estimates)
- and KL-divergence Hessian-vector product function $f(v) = \hat{H}_k v$

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Estimate proposed step $\Delta_k \approx \sqrt{\frac{2\delta}{x_k^T \hat{H}_k x_k}} x_k$

Perform backtracking line search with exponential decay to obtain final update

$$\theta_{k+1} = \theta_k + \alpha^j \Delta_k$$

end for

Proximal Policy Optimization

Can I achieve similar performance without second order information (no Fisher matrix!)

- Adaptive KL Penalty
  - Policy update solves unconstrained optimization problem

$$\theta_{k+1} = \arg\max_{\theta} \mathcal{L}_{\theta_k}(\theta) - \beta_k \bar{D}_{KL}(\theta || \theta_k)$$

  - Penalty coefficient β_k changes between iterations to approximately enforce KL-divergence constraint
- Clipped Objective
  - New objective function: let $r_t(\theta) = \pi_\theta(a_t|s_t)/\pi_{\theta_k}(a_t|s_t)$. Then

$$\mathcal{L}^{CLIP}_{\theta_k}(\theta) = \mathop{\mathrm{E}}_{\tau \sim \pi_k}\left[\sum_{t=0}^{T}\left[\min(r_t(\theta)\hat{A}_t^{\pi_k}, \text{clip}\left(r_t(\theta), 1-\epsilon, 1+\epsilon\right)\hat{A}_t^{\pi_k})\right]\right]$$

  where ϵ is a hyperparameter (maybe $\epsilon = 0.2$)
  - Policy update is $\theta_{k+1} = \arg\max_\theta \mathcal{L}^{CLIP}_{\theta_k}(\theta)$

J. Schulman, F. Wolski, P. Dhariwal, A. Radford, and O. Klimov. "Proximal Policy Optimization Algorithms". (2017)

PPO: Adaptive KL Penalty

Input: initial policy parameters θ_0, initial KL penalty β_0, target KL-divergence δ

for $k = 0, 1, 2, \ldots$ **do**

Collect set of partial trajectories $\mathcal{D}_k$ on policy $\pi_k = \pi(\theta_k)$

Estimate advantages $\hat{A}_t^{\pi_k}$ using any advantage estimation algorithm

Compute policy update

$$\theta_{k+1} = \arg\max_{\theta} \mathcal{L}_{\theta_k}(\theta) - \boxed{\beta_k \bar{D}_{KL}(\theta || \theta_k)}$$

by taking K steps of minibatch SGD (via Adam)

if $\bar{D}_{KL}(\theta_{k+1} || \theta_k) \geq 1.5\delta$ **then**

$\beta_{k+1} = 2\beta_k$

else if $\bar{D}_{KL}(\theta_{k+1} || \theta_k) \leq \delta/1.5$ **then**

$\beta_{k+1} = \beta_k / 2$

end if

end for

Don’t use second order approximation for Kl which is expensive, use standard gradient descent

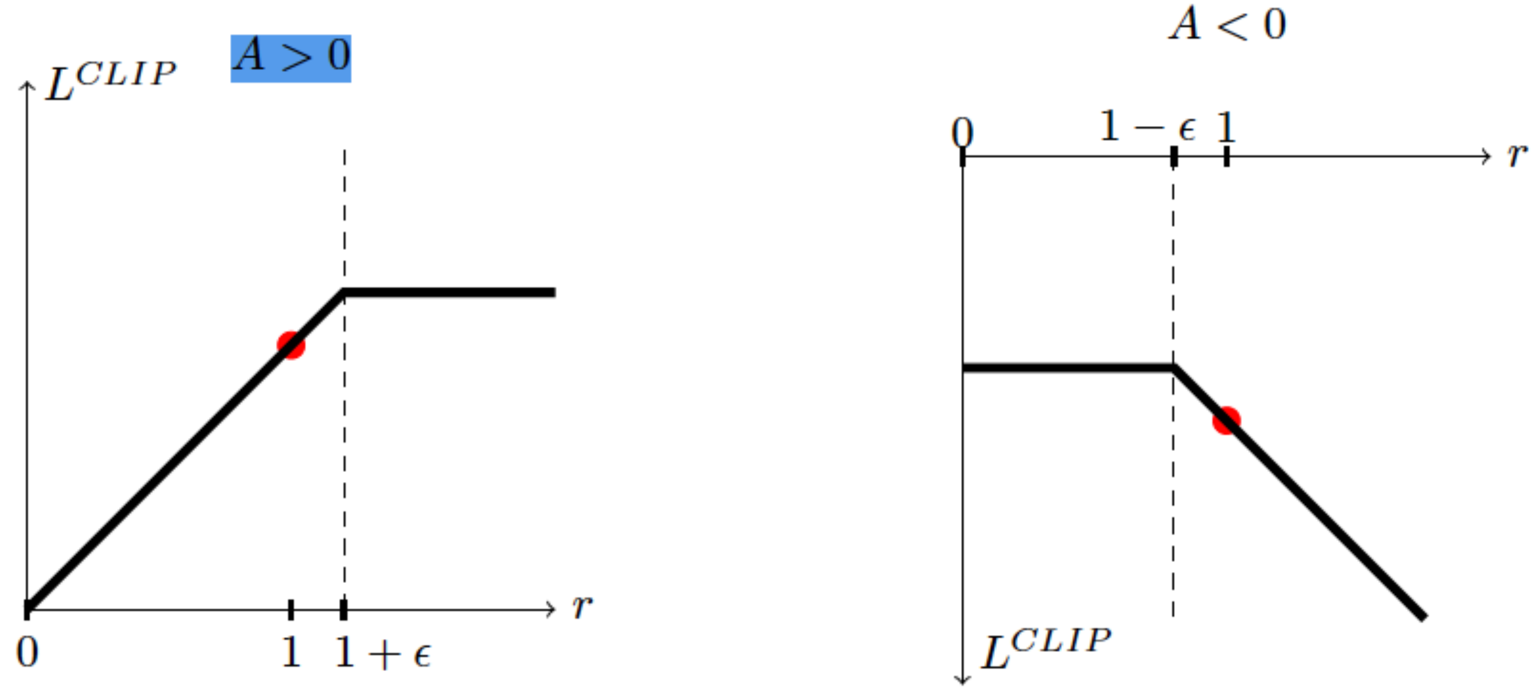
PPO: Clipped Objective

- Recall the surrogate objective

$$L^{IS}(\theta) = \hat{\mathbb{E}}_t\left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{\text{old}}}(a_t \mid s_t)}\hat{A}_t\right] = \hat{\mathbb{E}}_t\left[r_t(\theta)\hat{A}_t\right]. \tag{1}$$

- Form a lower bound via clipped importance ratios

$$L^{CLIP}(\theta) = \hat{\mathbb{E}}_t\left[\min(r_t(\theta)\hat{A}_t, \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_t)\right] \tag{2}$$



J. Schulman, F. Wolski, P. Dhariwal, A. Radford, and O. Klimov. "Proximal Policy Optimization Algorithms". (2017)

PPO: Clipped Objective

But *how* does clipping keep policy close? By making objective as pessimistic as possible about performance far away from θ_k:

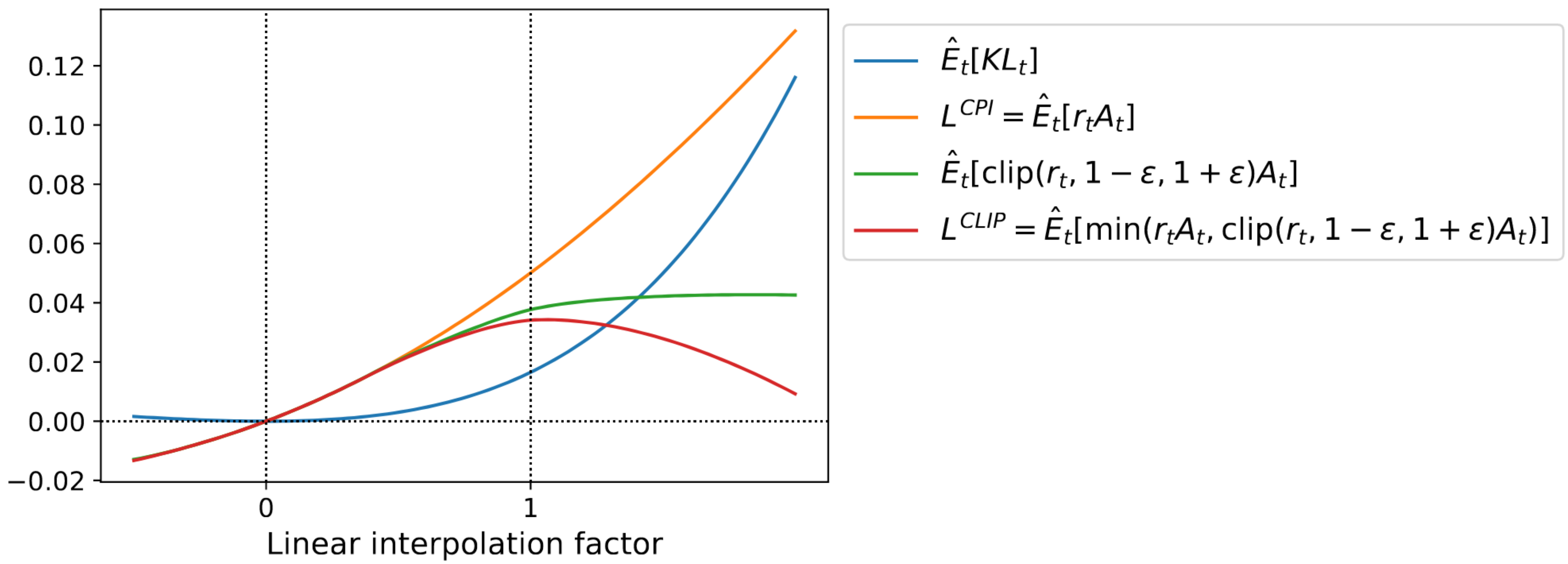


Figure: Various objectives as a function of interpolation factor α between θ_{k+1} and θ_k after one update of PPO-Clip [9]

PPO: Clipped Objective

Input: initial policy parameters θ_0, clipping threshold ϵ

for $k = 0, 1, 2, \ldots$ **do**

Collect set of partial trajectories $\mathcal{D}_k$ on policy $\pi_k = \pi(\theta_k)$

Estimate advantages $\hat{A}_t^{\pi_k}$ using any advantage estimation algorithm

Compute policy update

$$\theta_{k+1} = \arg\max_{\theta} \mathcal{L}_{\theta_k}^{CLIP}(\theta)$$

by taking K steps of minibatch SGD (via Adam), where

$$\mathcal{L}_{\theta_k}^{CLIP}(\theta) = \mathop{\mathrm{E}}_{\tau \sim \pi_k}\left[\sum_{t=0}^{T}\left[\min(r_t(\theta)\hat{A}_t^{\pi_k}, \mathrm{clip}\left(r_t(\theta), 1-\epsilon, 1+\epsilon\right)\hat{A}_t^{\pi_k})\right]\right]$$

end for

- Clipping prevents policy from having incentive to go far away from θ_{k+1}
- Clipping seems to work at least as well as PPO with KL penalty, but is simpler to implement

PPO: Clipped Objective

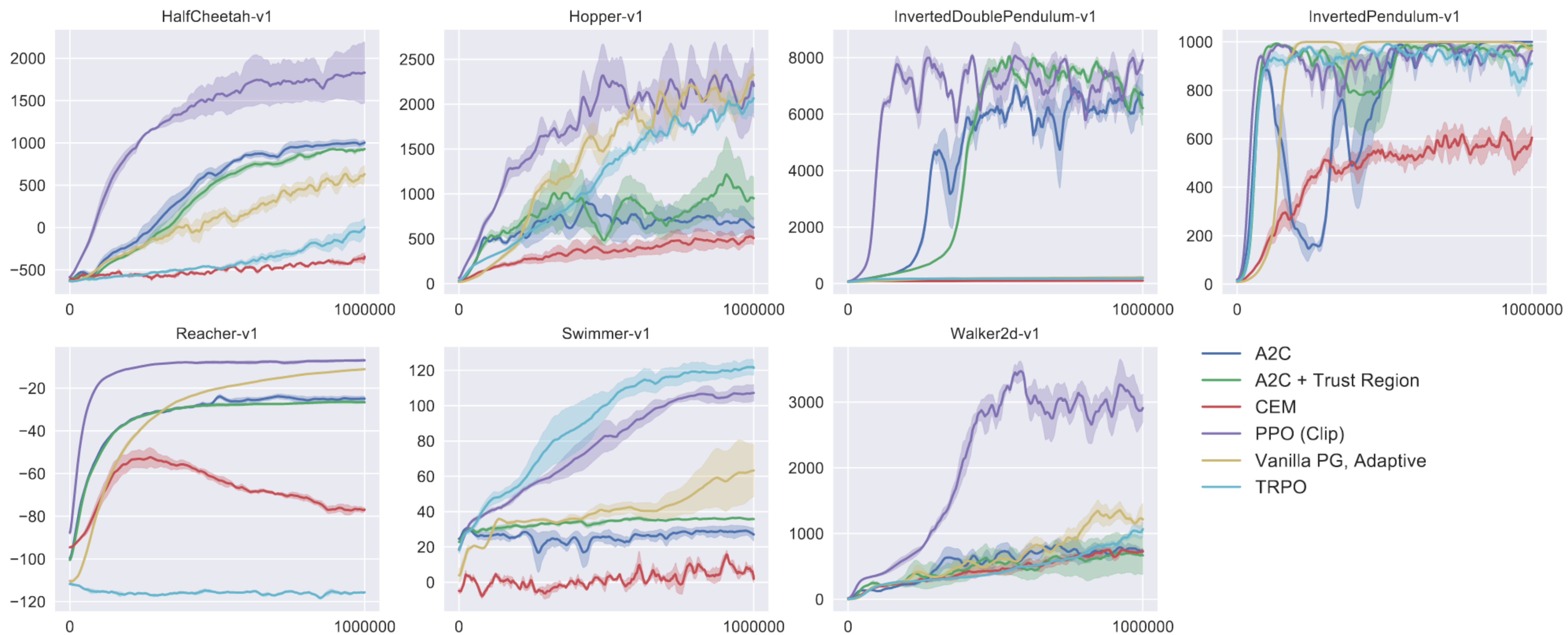


Figure: Performance comparison between PPO with clipped objective and various other deep RL methods on a slate of MuJoCo tasks. [10]

Summary

- Gradient Descent in Parameter VS distribution space
- Natural gradients: we need to keep track of how the KL changes from iteration to iteration
- Natural policy gradients
- Clipped objective works well