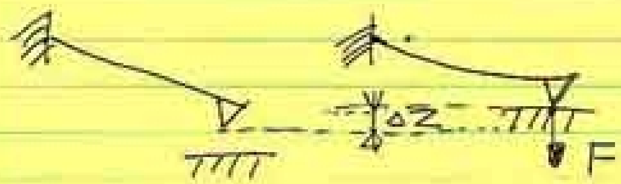
Mechanical Properties of Cantilevers

The AFM probe consists of an ultra-sharp tip mounted on the flexible beam (cantilever). The role of the cantilever is to translate the force acting on the tip into a measurable deflection.



It turns out, that the force and cantilever deflection are related through a simple linear relationship

$$F = k \, \Delta z$$

where k is a spring constant of a cantilever.

In this lecture we will focus on deriving this relationship and on determining how k depends on such properties of a cantilever as its size, shape and material.

Center of Mass

System of N particles

$$\vec{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r_i} \qquad M = \sum_{i=1}^{N} m_i$$

m_i - mass of an individual particle

r_i - particle position

Bulk body

$$\boxed{\vec{R} = \frac{1}{M}\iiint \rho\, \vec{r}\, dV}$$

(The integrals can be extended over all space, since $\rho = 0$ outside the body)

$\rho(\vec{r})$ - density

M - mass of a body

if density is uniform

$$\boxed{\vec{R} = \frac{1}{V}\iiint \vec{r}\, dV}$$

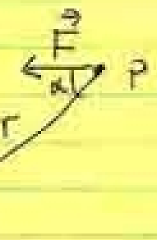
The motion of the center of mass is determined by the linear momentum theorem:

$$M\ddot{\vec{R}} = \vec{F}$$

$\ddot{\ }$ - second derivative

$\vec{F}$ - total external force

Rotational Motion



- Moment of force (torque) with respect to point O

$$\boxed{\vec{N_o} = \vec{r} \times \vec{F}} \qquad (N_o = r F \sin\alpha)$$

- Angular momentum

$$\vec{L_o} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

- Moment of inertia with respect to any axis (e.g. z)

$$\boxed{I_z = \sum_{i=1}^{N} m_i r_i^2} \quad - \quad N\text{-particles}$$

$$\boxed{I_z = \iiint \rho r^2 dV} \quad - \quad \text{bulk body}$$

Radius of gyration k_z

$$M k_z^2 = I_z$$

Moment of inertia describes the distribution of mass within the body and is a rotational analog of mass

Moment of Inertia as Rotational Analog of Mass

Consider kinetic energy T.
In rectilinear motion

$$T = \frac{1}{2} m v^2$$

In rotational motion

$$T = \frac{1}{2} I \omega^2$$

Show for a thin ring r_i consisting of individual particles m_i each

for an individual particle

$$T_i = m_i \frac{v_i^2}{2} = \frac{m_i}{2}\left(\frac{2\pi r_i}{T}\right)^2$$

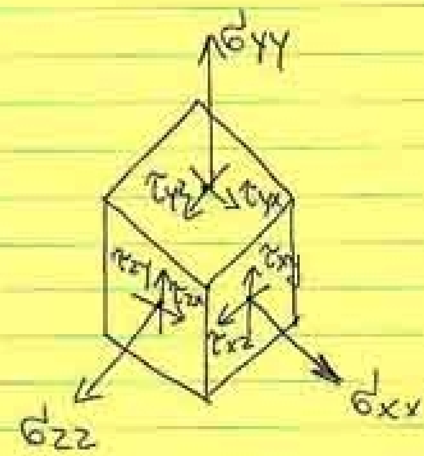
$$\frac{2\pi}{T} = \omega$$

$$T_i = \frac{m_i \omega^2 r_i^2}{2}$$

$$T_{total} = \frac{\omega^2}{2} \sum_{i=1}^{N} m_i r_i^2 =$$

$$= \frac{1}{2} I_z \omega^2$$

Stress and Strain



Consider a force $\vec{F}$ acting on a three-dimensional body in a rectangular coordinate system

For an element of area A_x perpendicular to the x direction we can identify three stress components:

normal stress: (axial) $\sigma_{xx} = \lim_{A_x \to 0} \frac{F_x}{A_x}$

shear stresses: $\tau_{xy} = \lim_{A_x \to 0} \frac{F_y}{A_x}$

$\tau_{xz} = \lim_{A_x \to 0} \frac{F_x}{A_x}$

(Stress components for [illegible] are defined)

Thus, a vectorial field of force (F_x, F_y, F_z) gives rise to a tensorial field of stress

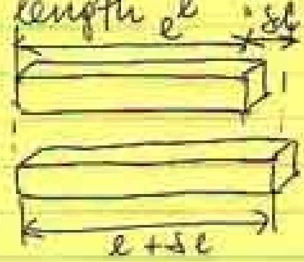
$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} .$$

In response to two types of stress (axial and shear), the body will experience a deformation that can be described by two types of strain (for simplicity we drop tensorial notation):

<u>Axial strain</u>

Describes the relative elongation δl of a bar of length l :

$$\boxed{\epsilon = \frac{\delta l}{l}}$$

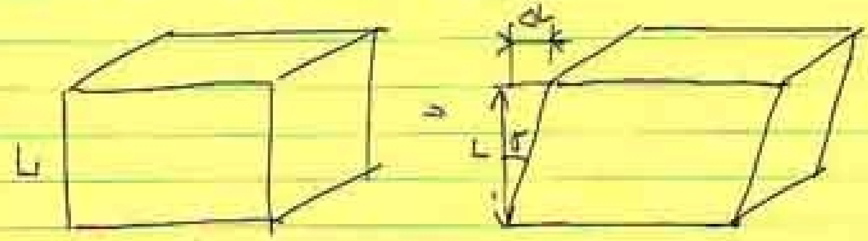


Within elastic regime the axial stress and strain are related by

$$\boxed{\sigma = E \epsilon}$$, where E - Young's modulus

Shear strain

The relative deformation in the direction perpendicular to the direction of the stress.



$$\boxed{\text{shear strain} = \frac{\Delta L}{L} = \tan\gamma = \gamma}$$

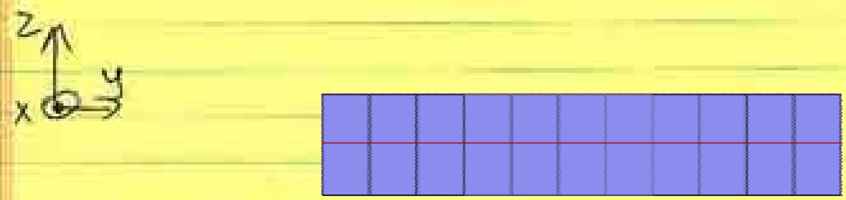
In the elastic regime:

$$\boxed{\tau = G\gamma}$$

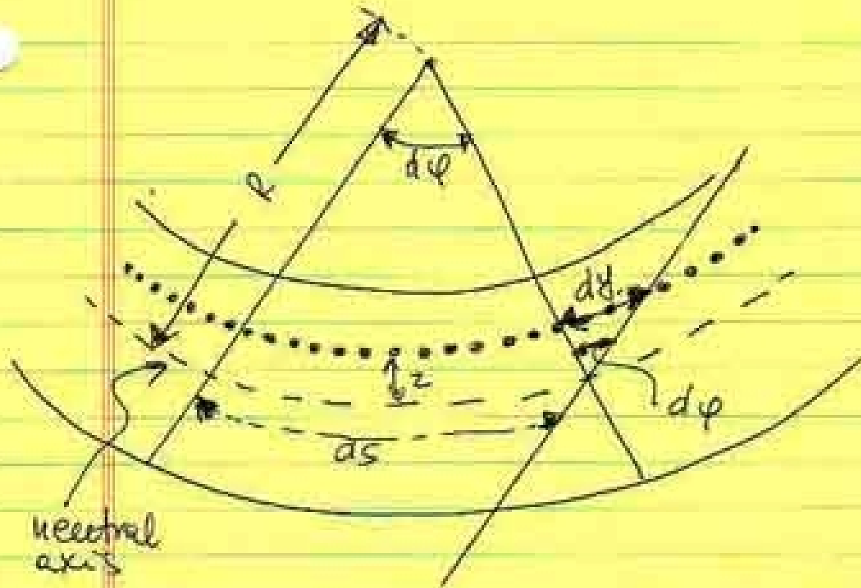
G - shear modulus of elasticity (or modulus of rigidity)

Balance of Moments in Bending Beam

Consider a beam bending around the x axis perpendicular to the surface of this page:



We realize that bending leads to compression and dilation in parts of a beam positioned respectively above and below the <u>neutral line</u> passing through the center of the beam.



Let us consider the deformation dy of a "fiber" shown by ••••, positioned at a distance z from the neutral axis.

The angle of bending, $d\varphi$ is related to the undeformed length ds and radius of curvature R by

$$d\varphi = \frac{dS}{R}$$

(for small values of $d\varphi$, $\frac{ds}{R} = \sin d\varphi = d\varphi$)

The deformation $dy = z\, d\varphi$ (by similar argument)

Thus, the strain in the "fiber" is equal to

$$\epsilon = -\frac{dy}{ds} = -\frac{z}{R}.$$

The corresponding stress

$$\sigma = \frac{dF}{dA} = E\epsilon = -E\frac{z}{R}$$

where E is the Young's modulus of the beam.

The total axial force acting on the cross-section perpendicular to the neutral axis is equal to

$$F_a = \iint_A dF = -\frac{E}{R}\iint_A z\,dA$$

Since upon bending there is no net change in length of the beam,

$$F_a = 0$$

This implies that the neutral layer contains the center of mass of the cross-sectional area of the beam (why?)

(see the definition of the center of mass)

Now let's consider the total torque (bending moment) exerted by forces dF on the cross-section A

$$M_{int} = \iint_A \underbrace{z\,dF}_{} =$$

subscript int indicates that there are internal torques

The torque produced by dF with respect to the axis of rotation passing through neutral axis and perpendicular to it.

$$= \iint_A \underbrace{\frac{dF}{dA}}_{\sigma} z\,dA = -\iint_A \frac{Ez}{R} z\,dA =$$

$$= -\frac{E}{R} \underbrace{\iint_A z^2 dA}_{} =$$

$$= \frac{-E}{R} I$$

Area moment of inertia

It will become clear in a moment that it is useful to rewrite this as

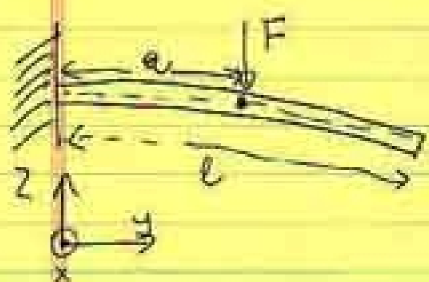
$$\frac{1}{R} = - \frac{M_{int}}{E I}$$

In equilibrium the internal torques are in balance with the external torque measured with respect to the same axis

$$M_{int} + M_{ext} = 0$$

Thus

$$\frac{1}{R} = \frac{M_{ext}}{E I} .$$



In order to proceed further, we use the expression for the curvature of a plane curve $z(y)$

$$\frac{1}{R} = \frac{d^2z/dy^2}{\left[1+\left(\frac{dz}{dy}\right)^2\right]^{3/2}} ,$$

which for small curvature is reduced to

$$\frac{1}{R} = \frac{d^2z}{dy^2} ,$$

and thus we obtain the differential equation describing the balance of torques

$$\frac{d^2z}{dy^2} = \frac{M_{ext}}{EI} .$$

The external torque produced by the force F acting at a distance "a" from the point of support of the cantilever, calculated with respect to any rotation axis positioned at a distance $y < a$ from the point of support

$$M_{ext} = F(y-a) ,$$

Thus we have to solve

$$\frac{d^2z}{dy^2} = \frac{F}{EI}(y-a) .$$

The first integration gives

$$\frac{dz}{dy} = \frac{1}{2}\frac{F}{EI}y^2 - \frac{F}{EI}ay + C_1$$

from the boundary condition

$$\frac{dz}{dy} = 0 \quad \text{for } y=0$$

we obtain $C_1 = 0$.

The second integration gives

$$z = \frac{Fy^2}{6EI}(y-3a) + C_2 .$$

Again, from the boundary condition

$z(y) = 0$ for $y=0 \Rightarrow C_2 = 0$

Thus, if the force F is applied to the free end of the cantilever $(a = \ell)$, the z position of its free end $(y = \ell)$ is given by

$$z = -\frac{\ell^3}{3EI}\cdot F$$

We define the spring constant of a cantilever as:

$$k_c = \left|\frac{F}{z}\right|$$

and thus

$$\boxed{k_c = 3\,\frac{EI}{\ell^3}} \qquad I = \iint_A z^2\,dA$$

Example

Calculate the spring constant of a cylindrical cantilever machined out of silicon. The length and radius of a cantilever are respectively 100 μm and 5 μm. Young's modulus of silicon is equal to 180 GPa

First we need to calculate the reduced radius of gyration of a circular cross-section of a cantilever with respect to the bending axis

Now we can use the <u>perpendicular axis theorem</u>, which states that:

For a planar object the moment of inertia about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two perpendicular axes passing through the same point in the plane of the object

$$I_z = I_x + I_y$$

Simple justification

$\Delta I_x = \Delta m\, x^2$

$\Delta I_y = \Delta m\, y^2$

$\Delta I_x + \Delta I_y =$
$= \Delta m (x^2 + y^2) =$
$= \Delta m\, r^2 = \Delta I_z$

Due to the symmetry of a circle the moment we are seeking

$$I = \pi \frac{R^4}{4}$$

Thus,

$$k_c = \frac{3 \cdot 180 \times 10^9 \frac{N}{m^2} \cdot \frac{\pi \cdot \left(5 \times 10^{-6} m\right)^4}{4}}{\left(100 \times 10^{-6} m\right)^3} =$$

$$\approx 220 \frac{N}{m}$$