<u>06-713: Homework 8</u>
Due Monday December 3

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1. This problem concerns the forced wave equation on an infinite domain,

$$z_{tt} = c^2 z_{xx} + F(x,t), \qquad -\infty < x < \infty, \qquad t > 0,$$
$$z(x,0) = f(x), \quad z_t(x,0) = g(x), \quad -\infty < x < \infty.$$

(a) Show that the solution to this PDE can be written as *z=v+w*, where *w* satisfies the unforced wave equation with the initial data above and *v* satisfies the forced wave equation with homogeneous initial data (that is *f(x)=g(x)=0* for all *x*).

(b) Write down an explicit expression for *w(x,t)*.

(c) The PDE for *v(x,t)* can be solved by

$$v(x,t) = \frac{1}{2c}\int_0^t dt' \int_{x-c(t-t')}^{x+c(t-t')} F(x',t')dx'.$$

Sketch the domain of dependence of the solution to the full forced problem at an arbitrary point, (x,t).

(d) Solve the full forced problem with *f(x)=g(x)=0* for all *x* and *F(x,t)*=exp*(-at)*, where *a* is a constant. Sketch the solution.

2. In class we discussed the wave equation for an undamped string pinned at both ends. In reality, the vibration of elastic strings is damped over time as energy is lost to the surroundings. We will asssume that $c > kL$. This problem can be modeled by

$$z_{tt} + kz_t = c^2 z_{xx}, \quad 0 < x < L, \; t > 0,$$
$$z(x,0) = f(x), \quad z_t(x,0) = g(x), \quad 0 < x < L,$$
$$z(0,t) = z(L,t) = 0, \quad t \ge 0.$$

(a) Find the general solution to this problem. [Hint: When solving the ODE for T(t), seek a solution of the form $e^{i\omega t}$.]

(b) Sketch the solution at $x = L/4$, $x = L/2$, and $x = 3L/4$ when $g(x) = 0$ and $f(x) = \sin(\pi x/L)$.

3. (Exam problem from 2006) Find the solution of

$$\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 u}{\partial \theta^2} = 0$$

outside the circle $r = a$ when $u(a,\theta) = f(\theta), 0 \le \theta < 2\pi$. Assume that $u(r,\theta)$ is bounded for all $r > a$.

(b) Plot the solution of the PDE above for $u(r,\theta=0)$ and $u(r=3a,\theta)$ when

$$f(\theta)=\begin{cases}\theta \text{ for } 0\le\theta\le\pi/2\\ \pi-\theta \text{ for } \pi/2\le\theta\le\pi\\ 0 \text{ for } \pi\le\theta\le 2\pi\end{cases}$$

Optional Problems

1. (Exam problem from 2006) Sound waves in a plasma are described by a set of three nonlinear PDES:

$$\phi_{xx}=e^{\phi}-\rho,\quad \rho_t+(\rho v)_x=0, \text{ and } v_t+vv_x=-\phi_x,$$

where ϕ is the electrical potential, ρ is the ion density, and v is the ion velocity. The problem has been assumed to be one dimensional and has been scaled so that all the relevant physical constants equal 1.

(a) Show that one solution of these PDEs is $\phi=0$, $\rho=1$, and $v=0$.

(b) Linearize the PDEs above by assuming that disturbances around the solution in (a) are small. Let $\rho=1+u$, and show that u satisfies $u_{ttxx}=u_{tt}-u_{xx}$.

(c) Solve the PDE from (b) with boundary conditions $u(0,t)=u(L,t)=0$ for $t>0$ and initial data $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$.

2. Greenberg Problem 18.3.29
3. Greenberg Problem 19.4.7
4. Greenberg Problem 19.4.12
5. Find the solution for the Laplace equation in a square, $u_{xx}+u_{yy}=0$ for $0\le x\le 1$ and $0\le y\le 1$ with the boundary conditions $u=0$ along all boundaries except $u(x,0)=f(x)$ for $0\le x\le 1$. You must give complete definitions of any expansion coefficients you use, but you can leave them written as integrals.