Homework 6 — solutions 06-713

1. (a)

$$x_p(t) = \frac{F}{m\left(\frac{k}{m}-\omega^2\right)} \cos(\omega t)$$

$$\frac{dx_p(t)}{dt} = \frac{-F\omega}{m\left(\frac{k}{m}-\omega^2\right)} \sin(\omega t)$$

$$\frac{d^2x_p(t)}{dt} = \frac{-F\omega^2}{m\left(\frac{k}{m}-\omega^2\right)} \cos(\omega t)$$

Substituting into the equation,

$$LHS = m\left[\frac{-F\omega^2}{m\left(\frac{k}{m}-\omega^2\right)} \cos(\omega t)\right] + k\left[\frac{F}{m\left(\frac{k}{m}-\omega^2\right)} \cos(\omega t)\right]$$

$$= F\cos\omega t\left[\frac{-m\omega^2}{k-m\omega^2} + \frac{k}{k-m\omega^2}\right] = F\cos\omega t = RHS.$$

Thus, x_p is a particular solution of the ODE.

(b) For general solution, first, we will solve the homogeneous equation.

$$m\frac{d^2x}{dt^2} + kx = 0.$$

Let the solution be the form $x(t) = e^{\lambda t}$

Substituting this to the equation,

$$m\lambda^2 + k = 0.$$

$$\lambda = \pm i\sqrt{k/m}$$

Hence, the solution is $X(t) = Ae^{i\sqrt{k/m}\,t} + Be^{-i\sqrt{k/m}\,t}$

or equivalently $X(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$

where A and B are unknown constants.

$\therefore$ General solution is

$$X(t) = \frac{F}{m\left(\frac{k}{m}-\omega^2\right)}\cos\omega t + A\cos\left(\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

(c) For uniqueness of this BVP, Consider the Fredholm alternative

$\therefore$ The BVP has a unique solution iff the homogeneous ODE with homogeneous boundary conditions has only the trivial solution $X(t)=0$.

Consider homogeneous BCs $X(0)=0$ and $X(T)=0$.

We will apply this to the homogeneous solution obtained in part (b).

$$X(t) = A\cos\left(\sqrt{k/m}\,t\right) + B\sin\left(\sqrt{k/m}\,t\right)$$

$X(0) = 0 \quad \Rightarrow \quad A = 0$

$X(T) = 0 \quad \Rightarrow \quad B\sin\left(\sqrt{\frac{k}{m}}T\right) = 0$

$\Rightarrow \quad B = 0 \quad$ iff $\quad \sqrt{\frac{k}{m}}T \neq n\pi \qquad n=1, 2, \ldots$

or $\quad T \neq \frac{n\pi}{\sqrt{k/m}} \qquad n=1, 2, \ldots$

Note that if $T = n\pi/\sqrt{k/m}$, we will not be able to find value of B even if we consider general solution.

Thus, system has unique solution if $T \neq \frac{n\pi}{\sqrt{k/m}} \quad n=1, 2, \ldots$

2. (a) $y'' - \mu^2 y = -Q_0(x)/k$

The homogeneous form of this ODE

: $y'' - \mu^2 y = 0.$

Assume $y = e^{mx}$

then, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

Substituting into the equation,

$$m^2 e^{mx} - \mu^2 e^{mx} = 0.$$

$$\Rightarrow m^2 = \mu^2$$

$$\therefore m = \pm\mu$$

General solution of the homogeneous ODE

$\Rightarrow$ $y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$

Applying BCs,

$y(0) = 0 \rightarrow y(0) = C_1 + C_2 = 0. \rightarrow C_2 = -C_1$ — ①.

$y'(L) = 0 \rightarrow \mu C_1 e^{\mu L} - \mu C_2 e^{-\mu L} = 0$

by ① $\hookrightarrow \mu C_1 (e^{\mu L} + e^{-\mu L}) = 0.$

$$\therefore C_1 = 0$$

$$\rightarrow C_2 = 0.$$

$\therefore y(x) = 0$ (trivial solution)

By Fredholm alternative, the equation $y'' - \mu^2 y = -Q_0(x)/k$ has a unique solution.

(b) In the ODE $y'' - \mu^2 y = -Q_0(x)/k$, linearly independent Solutions are given by $\phi_1 = e^{\mu x}$, $\phi_2 = e^{-\mu x}$

$$W(t) = \det\begin{bmatrix} e^{\mu t} & e^{-\mu t} \\ \mu e^{\mu t} & -\mu e^{-\mu t} \end{bmatrix} = -2\mu$$

$$\tilde{W}(x,t) = \det\begin{bmatrix} e^{\mu t} & e^{-\mu t} \\ e^{\mu x} & e^{-\mu x} \end{bmatrix} = e^{\mu(t-x)} - e^{-\mu(t-x)}$$

∴ green function

$$G(x,t) = \frac{\tilde{W}(x,t)}{a_0(t)\,W(n)} = \frac{e^{\mu(t-x)} - e^{-\mu(t-x)}}{1 \cdot (-2\mu)}$$

$$= \frac{e^{\mu(t-x)} - e^{-\mu(t-x)}}{-2\mu}$$

then, $y_p(x) = -\int_{x_0}^{x} \frac{Q_0(t)}{k}\left(\frac{e^{\mu(t-x)} - e^{-\mu(t-x)}}{-2\mu}\right) dt$

$$= \int_0^L \frac{Q_0(t)}{k}\,\frac{e^{\mu(t-x)} - e^{-\mu(t-x)}}{2\mu}\, dt$$

From part (a), $y_h(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$\therefore\ y(x) = y_h(x) + y_p(x)$

$$= c_1 e^{\mu x} + c_2 e^{-\mu x} + \int_0^L \frac{Q_0(t)}{k}\,\frac{e^{\mu(t-x)} - e^{-\mu(t-x)}}{2\mu}\, dt$$

with $y(0) = 0$, $y'(L) = 0$.

Temperature profile : $T(x) = T_0 + y(x)$

The solution is physically consistent because temperature will increase along the length of rod.

(c) Green function is independent of the inhomogeneous part of ODE. Therefore, the green function would not be changed when $Q(x) = Q_0 \sin(x)$.

When the rod are insulated at both ends, green function would not be changed. But at this time, boundary conditions will change $y'(0) = 0$ & $y'(L) = 0$

3. (b)

$y'' + (\cos x)\, y = 0. \qquad y(0) = 1, \quad y(10) = 2.$

$\rightarrow \quad y' = u \qquad y(0) = 1, \quad y = (10) = 2$

$u' + (\cos x)\, y = 0.$

We can write the system of IVP.

$L(y_1) = 0 \qquad y_1(0) = 1, \qquad y_1'(0) = 0$

$L(y_2) = 0 \qquad y_2(0) = 0 \qquad y_2'(0) = 1$

In this case, since the equation is homogeneous,

$y_p(x) = 0.$

$L = \dfrac{d^2}{dx^2} - \cos x$, differential operator.

The general solution can be written as

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

Using the conditions,

$$y(0) = c_1 y_1(0) + c_2 y_2(0) \Rightarrow c_1 = 1$$

$$y(10) = c_1 y_1(10) + c_2 y_2(10) = 2 \Rightarrow y_1(10) + c_2 y_2(10) = 2$$

$$\rightarrow \quad c_2 = \frac{2 - y_1(10)}{y_2(10)}$$

Substituting c_1 and c_2 into the equation,

$$y(x) = y_1(x) + \left\{ \frac{2 - y_1(10)}{y_2(10)} \right\} y_2(x)$$

$$y_1' = u_1 \qquad y_1(0) = 1$$

$$u_1' + (\cos x) y_1 = 0 \qquad u_1(0) = 0.$$

$$y_2' = u_2 \qquad y_2(0) = 0$$

$$u_2' + (\cos x) y_2 = 0 \qquad u_2(0) = 1$$

These equations can be solved using mathematical software packages.

$$y_1(2) = -0.21714 \qquad y_1(10) = 42.584$$

$$y_2(2) = 1.400165 \qquad y_2(10) = -77.2165$$

$$\therefore \quad y(2) = y_1(2) + \left\{ \frac{2 - y_1(10)}{y_2(10)} \right\} y_2(2)$$

$$= -0.21714 + \frac{2 - 42.584}{-77.2165} \times 1.400165$$

$$= \underline{0.518769}$$

(e) $y''' + xy = 2x^3$

$y(1) = y'(1) = 0 \quad , \qquad y''(2) = -3.$

$$y' = u \quad , \qquad y(1) = 0$$
$$u' = v \quad , \qquad u(1) = 0$$
$$v' = -xy + 2x^3 \quad , \qquad v(2) = -3$$

When $L = \dfrac{d^3}{dx^3} + x$, differential operator ,

$L[y_1] = 0$		$y_1(1) = 1$	$y_1'(1) = 0$	$y_1''(1) = 0$
$L[y_2] = 0$		$y_2(1) = 0$	$y_2'(1) = 1$	$y_2''(1) = 0$
$L[y_3] = 0$		$y_3(1) = 0$	$y_3'(1) = 0$	$y_3''(1) = 1$
$L[y_p] = 2x^3$		$y_p(1) = 0$	$y_p'(1) = 0$	$y_p''(1) = 0.$

The general solution can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + y_p(x)$$

$$y(1) = c_1 \cdot 1 = 0 \quad \rightarrow \quad c_1 = 0$$

$$y'(1) = c_1 \cdot 0 + c_2 \cdot 1 + 0 + 0 = 0 \quad \rightarrow \quad c_2 = 0$$

$$y''(2) = 0 + 0 + c_3 y_3''(2) + y_p''(2) = -3$$

$$c_3 = \frac{-3 - y_p''(2)}{y_3''(2)}$$

$$\therefore \quad y(x) = \frac{-3 - y_p''(2)}{y_3''(2)} \cdot y_3(x) + y_p(x)$$

$$\left.\begin{aligned} y_3' &= u_3 & \qquad y_3(1) &= 0 \\ u_3' &= v_3 & u_3(1) &= 0 \\ v_3' &= -xy_3 & v_3(1) &= 1 \end{aligned}\right]$$

$$\left.\begin{aligned} y_p' &= u_p & \qquad y_p(1) &= 0 \\ u_p' &= v_p & u_p(1) &= 0 \\ v_p' &= -xy_p + 2x^3 & v_p(1) &= 0 \end{aligned}\right]$$

These equations can be solved using mathematical software packages.

$$y_3(2) = 0.48756 \qquad y_3''(2) = 0.71202$$

$$y_p(2) = 0.69316 \qquad y_p''(2) = 7.2246.$$

$$\therefore \; y(2) = \frac{-3 - 7.2246}{0.71202} \times 0.48756 + 0.69316$$

$$= \underline{-6.30819}$$

4. (a) Step 1 : Find the steady state.

$$\frac{dS}{dt} = 0, \quad \frac{dn_1}{dt} = 0, \quad \frac{dn_2}{dt} = 0.$$

$$0 = D(S_0 - S) - \frac{1}{Y_S}\frac{\mu_{S,max} S n_1}{K_S + S}$$

$$0 = -Dn_1 + \frac{\mu_{S,max} S n_1}{K_S + S} - \frac{1}{Y_P}\frac{\mu_{P,max} n_1 n_2}{K_P + n_1}$$

$$0 = -Dn_2 + \frac{\mu_{P,max} n_1 n_2}{K_P + n_1}$$

Using mathematical software packages, the steady states are given.

$\boxed{1}$: $S_s = 0.5$, $n_{1s} = 0$, $n_{2s} = 0$

$\boxed{2}$: $S_s = 1.67 \times 10^{-4}$, $n_{1s} = 1.65 \times 10^{-10}$, $n_{2s} = 3 \times 10^{-23}$

Step 2: In order to characterize the steady state, calculate the Jacobian matrix.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial n_1} & \frac{\partial f_1}{\partial n_2} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial n_1} & \frac{\partial f_2}{\partial n_2} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial n_1} & \frac{\partial f_3}{\partial n_3} \end{bmatrix}$$

At steady state $\boxed{1}$,

$$J = \begin{bmatrix} -0.0625 & -7.572 \times 10^{8} & 0 \\ 0 & 0.187375 & 0 \\ 0 & 0 & -0.0625 \end{bmatrix}$$

Eigenvalues : $\lambda_1 = 0.187375$

$\lambda_2 = \lambda_3 = -0.0625$

$\Rightarrow$ steady state : saddle point.

At steady state $\boxed{2}$

$$J = \begin{bmatrix} -140.64 & -1.89 \times 10^{8} & 0 \\ 4.64 \times 10^{-8} & 1.497 \times 10^{-14} & -7.04 \times 10^{-7} \\ 0 & 1.81 \times 10^{-16} & -0.0615 \end{bmatrix}$$

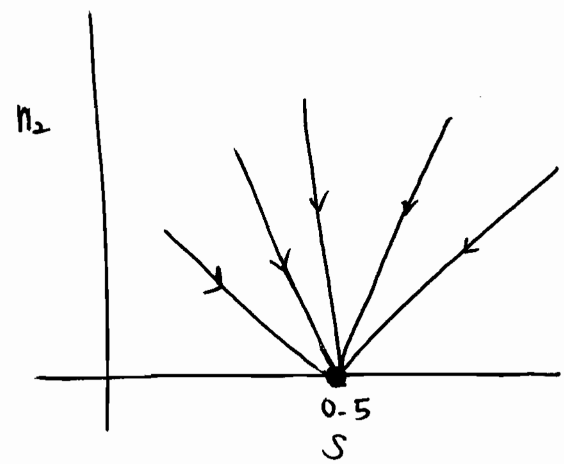
eigenvalues : $\lambda_1 = -140.578$

$\lambda_2 = -0.0625$

$\lambda_3 = -0.615$

all $\lambda_i < 0 \Rightarrow$ steady state : stable node

The trajectories on the plane $n_1 = 0$ converge to the steady state while other trajectories will diverge to other points.



$n_1 = 0$, the trajectories converge to the steady state.

b) $\frac{dS}{dt} = 0$, $S(t) = 0.5$.

$$\frac{dn_1}{dt} = -Dn_1 + \frac{\mu_{S,max}(0.5)n_1}{K_S + 0.5} - \frac{1}{Y_P}\frac{\mu_{P,max}\, n_1 n_2}{K_P + n_1}$$

$$\frac{dn_2}{dt} = -Dn_2 + \frac{\mu_{P,max}(n_1 n_2)}{K_P + n_1}$$

Steady states are given.

1. $n_{1s} = 0$, $n_{2s} = 0$.
2. $n_{1s} = 1.40845 \times 10^{-8}$, $n_{2s} = 5.90761 \times 10^{-5}$

At steady state $\boxed{1}$,

$$J = \begin{bmatrix} 0.1875 & 0 \\ 0 & -0.0625 \end{bmatrix}$$

eigenvalues : $\lambda_1 = 0.18725$, $\lambda_2 = -0.0625$

$$\lambda_1 > 0 > \lambda_2$$

$\Rightarrow$ Saddle point & <u>unstable</u>

At steady state $\boxed{2}$,

$$J = \begin{bmatrix} 0.048763 & -4.4 \times 10^{-5} \\ 193.8819 & 0 \end{bmatrix}$$

eigenvalues : $\lambda_1 = 0.02438 + 0.0897\, i$

$$\lambda_2 = 0.02438 - 0.0897\, i$$

$$Re(\lambda_1) = Re(\lambda_2) > 0$$

$\rightarrow$ <u>unstable</u> node

Thus if the initial condition is $n_1 = 0$, $n_2 > 0$,

then the trajectories move towards steady state $n_1 = 0$, $n_2 = 0$

Other than that, n_1 and n_2 increase infinitely with time.