06-713: Homework 5
Due Oct. 22

1. (a) Find the general solution to the equation $2xy\frac{dy}{dx} - y^2 = x^2$.

(b) Find the general solution to the equation $\frac{dy}{dx} = y + \sqrt{y}$.

(c) Solve the following initial value problem: $\frac{dy}{dx} = xy + x + y + 1, y(0)=0$. State the values of x for which the solution is valid.

2. A constant volume batch reactor undergoes the series reaction sequence $A \xrightarrow{k_1} B \xrightarrow{k_2} C$. Species A has an initial concentration of C_{A0}, and B and C are not present initially. The reaction rates per unit reactor volume are described by $R_A = -k_1 C_A$ and $R_B = k_1 C_A - k_2 C_B^2$.

(a) Write material balances for species A and B.

(b) Solve for C_A as a function of time.

(c) Show that the material balance for C_B takes the form of Riccati's equation when time is scaled using $\theta = k_2 t$.

(d) Use the Riccati transformation $C_B = \frac{1}{u}\frac{du}{d\theta}$ to transform the nonlinear first order equations from part (c) into a linear second order equation (you do not need to solve this equation). Is this second order equation autonomous?

(e) Find an explicit solution for the concentrations of all three species as a function of time is the reaction rate for B obeys first order kinetics, $R_B = k_1 C_A - k_2 C_B$.

3. This problem examines the analytical and numerical solutions of the system of ODEs $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = (x_1,...,x_n)^T$ and $\mathbf{A}$ is a diagonalizable (nxn) matrix of constants with eigenvalues, $\lambda_1,\ldots,\lambda_n$.

(a) The formal solution of these ODEs is $\mathbf{x}(t) = \mathbf{x}(0)\exp(\mathbf{A}t)$. Show that if $\mathbf{A}$ is negative definite, then $\mathbf{x}(t) \to \vec{0}$ for large t.

(b) Show that if Euler's method is used to numerically solve these ODEs, it yields a system of linear difference equations given by $\mathbf{x}_{n+1} = (\mathbf{I} + h\mathbf{A})\mathbf{x}_n$. If $\mathbf{A}$ is negative definite, what values of the time step, h, and eigenvalues, λ_i, give solutions that behave in the same way as the analytic solution from (a)?

(c) Determine an explicit form for the linear difference equations that arise from applying the implicit Trapezoidal method to the system of ODEs. Determine what values of the time step and eigenvalues give the correct qualitative behavior for large t if $\mathbf{A}$ is negative definite.

(d) For $\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 2 & -5 & 0 \\ -2 & 2 & -6 \end{bmatrix}$, determine via numerical calculations the maximum time step that gives the correct qualitative behavior for large t if Euler's method is used What is the maximum allowable time step using the 4th order Runge-Kutta method?

4. Find and classify the steady states of

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = -x_1 + (1 - x_1^2)x_2.$$

Also plot some representative trajectories in the phase plane numerically. What happens to trajectories at large times?

(b) The system of equations

$$\frac{dx_1}{dt} = -x_2 + x_1(1 - x_1^2 - x_2^2)^2,$$

$$\frac{dx_2}{dt} = x_1 + x_2(1 - x_1^2 - x_2^2)^2.$$

has a steady state at the origin. Describe the stability of this steady state. Also show that these equations have a periodic solution with x_1=cos t and x_2=sin t. Use numerically generated trajectories to examine the stability of this periodic solution.

Optional Problems

1. Many books contain large collections of problems for practicing the identification and solution of ODEs. One good source is section 2.11 of Boyce and DiPrima's "Elementary Differential Equations and Boundary Value Problems" (4th ed.). This source contains a mixture of ODEs that can be solved with standard techniques as well as some interesting applications. It would be a useful exercise to look at problems 1-32 in this section and to identify what methods you would use to solve them (but not necessarily actually solve every equation). Chapter 2 of Greenberg also contains many examples, but these are ordered by solution method, so they are less useful for testing your skills at identifying the various types of equations.
2. A very simple set of ODEs that exhibits chaos is the Rossler equations:

$$\frac{dx}{dt} = -(y + z)$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c).$$

Note that the only nonlinearity appears in the third equation. By numerical simulations, show that for a = b = 0.2 and c = 5.7 the solutions of these equations are chaotic. Also find all steady states of the equations and classify their stability.

3. Learn about what numerical methods Mathematica uses to numerically integrate ODEs. Experiment with how accurately Mathematica's methods work for stiff equations.