Homework 2 - Solutions

1. (a) We want to fit a quadratic polynomial

$$y = a + bt + ce^t$$

For a collection of m data points, using this equation, we will have

$$a + bt_i + ce^{t_i} = y_i \qquad \forall i = 1, 2, \cdots, m$$

or

$$\underbrace{\begin{bmatrix} 1 & t_1 & e^{t_1} \\ 1 & t_2 & e^{t_2} \\ \vdots & \vdots & \vdots \\ 1 & t_m & e^{t_m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{b}$$

For a least squares fit, we will minimize the sum of the squares of difference in actual and calculated value.

Solve the following minimization problem.

$$\min E^2 = \sum_{i=1}^{m} (a + bt_i + ce^{t_i} - y_i)^2$$

$$\Rightarrow \nabla E^2 = 0 \Leftrightarrow \begin{bmatrix} \frac{\partial E^2}{\partial a} \\ \frac{\partial E^2}{\partial b} \\ \frac{\partial E^2}{\partial c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2\sum_i (a + bt_i + ce^{t_i} - y_i) = 0$$

$$2\sum_i (a + bt_i + ce^{t_i} - y_i)t_i = 0$$

$$2\sum_i (a + bt_i + ce^{t_i} - y_i)e^{t_i} = 0$$

in matrix form

$$\begin{bmatrix} m & \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} e^{t_i} \\ \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2 & \sum_{i=1}^{m} t_i e^{t_i} \\ \sum_{i=1}^{m} e^{t_i} & \sum_{i=1}^{m} t_i e^{t_i} & \sum_{i=1}^{m} (e^{t_i})^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} y_i \\ \sum_{i=1}^{m} y_i t_i \\ \sum_{i=1}^{m} y_i e^{t_i} \end{bmatrix} \quad \text{--- (1)}$$

Now, since, $A^T A$ can be written as,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \\ e^{t_1} & e^{t_2} & \cdots & e^{t_m} \end{bmatrix} \begin{bmatrix} 1 & t_1 & e^{t_1} \\ 1 & t_2 & e^{t_2} \\ \vdots & \vdots & \vdots \\ 1 & t_m & e^{t_m} \end{bmatrix} = \begin{bmatrix} m & \sum_i t_i & \sum_i e^{t_i} \\ \sum_i t_i & \sum_i t_i^2 & \sum_i t_i e^{t_i} \\ \sum_i e^{t_i} & \sum_i t_i e^{t_i} & \sum_i (e^{t_i})^2 \end{bmatrix}$$

and $A^T b$ can be written as,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \\ e^{t_1} & e^{t_2} & \cdots & e^{t_m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \sum_i y_i \\ \sum_i y_i t_i \\ \sum_i y_i e^{t_i} \end{bmatrix}$$

then (1) can be written as

$$A^T A x = A^T b$$

Hence, the least squares solution $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ of this problem is given by the normal equations

$$A^T A x = A^T b.$$

(b) Note that the form of normal equations $A^T A x = A^T b$ is implicit.

x can be calculated explicitly by

$$x = (A^T A)^{-1} (A^T b) \quad : \text{explicit form}$$

In this problem, we just have to write the explicit form without deriving them.

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

The coefficient can be calculated from

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} m & \sum_{i=1}^m t_i & \sum_{i=1}^m t_i^2 & \cdots & \sum_{i=1}^m t_i^n \\ \sum_{i=1}^m t_i & \sum_{i=1}^m t_i^2 & & \cdots & \sum_{i=1}^m t_i^{n+1} \\ \vdots & & & & \\ \sum_{i=1}^m t_i^n & \sum_{i=1}^m t_i^{n+1} & & \cdots & \sum_{i=1}^m t_i^{2n} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m y_i t_i \\ \vdots \\ \sum_{i=1}^m y_i t_i^n \end{bmatrix}$$

(c) Ammonia is compressed in a piston-cylinder apparatus from an initial state of 30°C and 500 kPa to a final pressure of 1400 kPa. The following data were obtained during the process:

P/kPa	500	653	802	945	1100	1248	1400
V/L	1.25	1.08	0.96	0.84	0.72	0.60	0.50

Fit $\ln(P/\text{kPa})$ vs $\ln(V/L)$ to polynomials.

$y =$ $\ln(P/\text{kPa})$	6.215	6.482	6.687	6.685	7.003	7.129	7.244
$t =$ $\ln(V/L)$	0.223	0.077	−0.041	−0.174	−0.329	−0.511	−0.693

i) linear $\quad y = C + Dt$

$$\begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i t_i \end{bmatrix}$$

$$\begin{bmatrix} 7 & -1.448 \\ -1.448 & 0.937226 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 47.445 \\ -10.519296 \end{bmatrix}$$

$C = 6.54918$

$D = -1.10548$

$\therefore \; y = 6.54918 - 1.10548t$

ii) quadratic $\quad y = F + Gt + Ht^2$

$$\begin{bmatrix} m & \Sigma t_i & \Sigma t_i^2 \\ \Sigma t_i & \Sigma t_i^2 & \Sigma t_i^3 \\ \Sigma t_i^2 & \Sigma t_i^3 & \Sigma t_i^4 \end{bmatrix} \begin{bmatrix} F \\ G \\ H \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i t_i \\ \Sigma y_i t_i^2 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -1.448 & 0.937226 \\ -1.448 & 0.937226 & -0.435649522 \\ 0.937226 & -0.435649522 & 0.313966981 \end{bmatrix} \begin{bmatrix} F \\ G \\ H \end{bmatrix} = \begin{bmatrix} 47.445 \\ -10.519296 \\ 6.659600508 \end{bmatrix}$$

$F = 6.56998$

$G = -1.3789$

$H = -0.577777$

$\therefore \; y = 6.56998 - 1.3789t - 0.577777t^2$

iii) cubic $y = P + Qt + Rt^2 + St^3$

$$\begin{bmatrix} m & \Sigma t_i & \Sigma t_i^2 & \Sigma t_i^3 \\ \Sigma t_i & \Sigma t_i^2 & \Sigma t_i^3 & \Sigma t_i^4 \\ \Sigma t_i^2 & \Sigma t_i^3 & \Sigma t_i^4 & \Sigma t_i^5 \\ \Sigma t_i^3 & \Sigma t_i^4 & \Sigma t_i^5 & \Sigma t_i^6 \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \\ S \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i t_i \\ \Sigma y_i t_i^2 \\ \Sigma y_i t_i^3 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -1.448 & 0.939226 & -0.495649522 \\ -1.448 & 0.939226 & -0.495649522 & 0.313966981 \\ 0.939226 & -0.495649522 & 0.313966981 & -0.198135044 \\ -0.495649522 & 0.313966981 & -0.198135044 & 0.129987626 \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \\ S \end{bmatrix} = \begin{bmatrix} 47.445 \\ -10.519296 \\ 6.659600508 \\ -3.595319308 \end{bmatrix}$$

$P = 6.57389$

$Q = -1.37839$

$R = -0.696969$

$S = -0.171652$

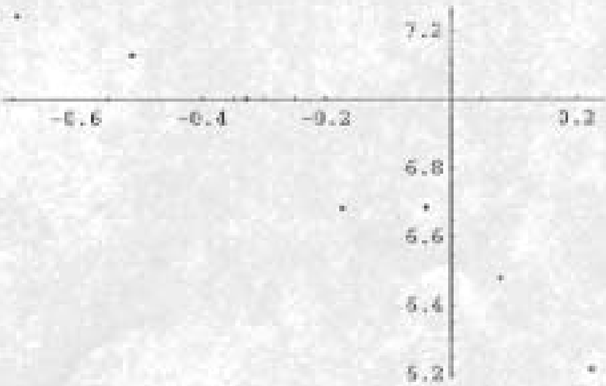
$\therefore\ y = 6.57389 - 1.37839t - 0.696969t^2 - 0.171652t^3$

```
<< Statistics`LinearRegression`
```

```
data = {{0.223, 6.215}, {0.077, 6.482}, {-0.041, 6.687},
  {-0.174, 6.685}, {-0.329, 7.003}, {-0.511, 7.129}, {-0.693, 7.244}}
```

```
{{0.223, 6.215}, {0.077, 6.482}, {-0.041, 6.687},
 {-0.174, 6.685}, {-0.329, 7.003}, {-0.511, 7.129}, {-0.693, 7.244}}
```

```
dplot = ListPlot[data, PlotStyle → {Hue[1]}]
```



- Graphics -

```
Regress[data, {1, x}, x]
```

{ParameterTable →

	Estimate	SE	TStat	PValue
1	6.54918	0.0373275	175.452	1.1612×10^{-10}
x	-1.10548	0.102013	-10.8366	0.000116146

RSquared → 0.959161, AdjustedRSquared → 0.950993, EstimatedVariance → 0.00663629,

ANOVATable →

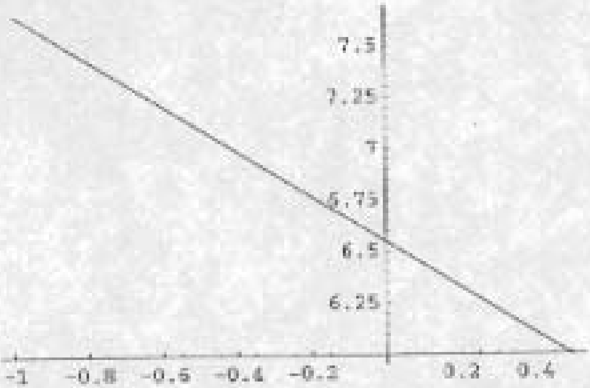
	DF	SumOfSq	MeanSq	FRatio	PValue
Model	1	0.779315	0.779315	117.432	0.000116146
Error	5	0.0331814	0.00663629		
Total	6	0.812497			

}

```
func = Fit[data, {1, x}, x]
```

y = 6.54918 - 1.10548 x → linear

```
Plot[{6.549181443130841` - 1.1054764489531228` x}, {x, -1, 0.5}, PlotStyle → {Hue[1]}]
```



- Graphics -

```
Regress[data, {1, x, x^2}, x]
```

		Estimate	SE	TStat	PValue
{ParameterTable →	1	6.56998	0.0326891	200.984	3.67651×10^{-9}
	x	-1.3789	0.170336	-8.09514	0.00126565
	x^2	-0.577777	0.313313	-1.84409	0.138938

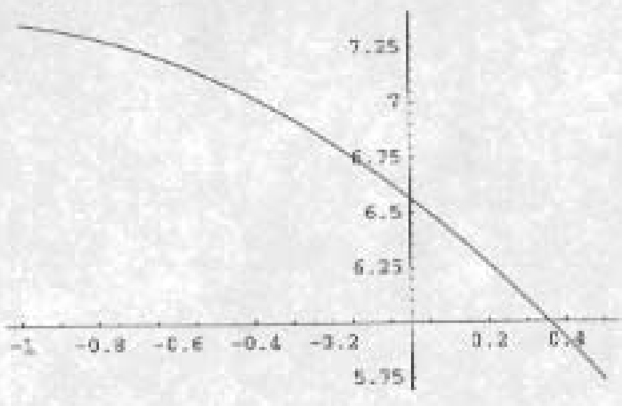
RSquared → 0.977927, AdjustedRSquared → 0.96689, EstimatedVariance → 0.00448358,

		DF	SumOfSq	MeanSq	FRatio	PValue
ANOVATable →	Model	2	0.794563	0.397281	88.608	0.000487222}
	Error	4	0.0179343	0.00448358		
	Total	6	0.812497			

```
func = Fit[data, {1, x, x^2}, x]
```

$y = 6.56998 - 1.3789x - 0.577777x^2$ → quadratic

```
Plot[{6.569980808976981` - 1.3788962945981231` x - 0.577776862162315` x^2},
 {x, -1, 0.5}, PlotStyle → {Hue[1]}]
```



- Graphics -

```
Regress[data, {1, x, x^2, x^3}, x]
```

		Estimate	SE	TStat	PValue
{ParameterTable →	1	6.57389	0.0503473	130.571	9.90468×10^{-7}
	x	-1.37839	0.196289	-7.02225	0.00593215
	x^2	-0.696968	1.08075	-0.644894	0.564918
	x^3	-0.171654	1.46707	-0.117004	0.91425

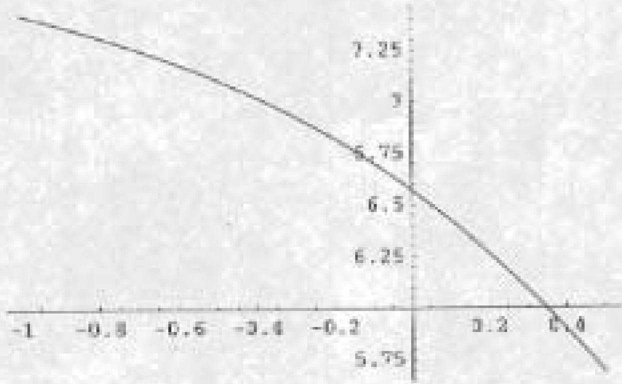
RSquared → 0.978027, AdjustedRSquared → 0.956054, EstimatedVariance → 0.00595095,

		DF	SumOfSq	MeanSq	FRatio	PValue
ANOVATable →	Model	3	0.794644	0.264881	44.5107	0.00548281}
	Error	3	0.0178529	0.00595095		
	Total	6	0.812497			

```
func = Fit[data, {1, x, x^2, x^3}, x]
```

y= $6.57389 - 1.37839\,x - 0.696968\,x^2 - 0.171654\,x^3$ ⟶ Cubic

```
Plot[{6.573890492856733` - 1.3783861350374176` x - 0.6969676673681668` x^2 -
   0.17165352737406717` x^3}, {x, -1, 0.5}, PlotStyle → {Hue[1]}]
```



- Graphics -

In both cases, we can get same results
(explicit solution and package's data fitting results)

2. (a) Show that $A^{-3}x = (1/\lambda)^3 x$

$$Ax = \lambda x$$
$$A^{-1}Ax = A^{-1}\lambda x$$
$$Ix = \lambda(A^{-1}x)$$
$$A^{-1}x = (1/\lambda)^1 x$$
$$A^{-1}A^{-1}x = A^{-1}(1/\lambda)^1 x$$
$$(A^{-1})^2 x = (1/\lambda)^2 x$$
$$\cdots \quad A^{-3}x = (1/\lambda)^3 x$$

$\therefore$ eigenvalues of A^{-3} : $(1/\lambda)^3$, where λ is eigenvalues of A
eigenvectors of A^{-3} : x , where x is eigenvectors of A

```
In[39]:= A = ( 1   1   3
              -2  -3   2
               3   1  -2 )
Out[39]= {{1, 1, 3}, {-2, -3, 2}, {3, 1, -2}}
In[40]:= λ1 = N[Eigenvalues[A]]
Out[40]= {-4.43008, 2.69313, -2.26305}
In[41]:= λ1^-3
Out[41]= {-0.0115018, 0.0511949, -0.0862816}
In[42]:= N[Eigenvectors[A]]
Out[42]= {{-0.234525, -1.72651, 1.}, {1.63923, -0.224562, 1.}, {-10.4047, 30.9511, 1.}}
```

(Theorem 11.4.2 If an $n \times n$ matrix A has distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, then the corresponding eigenvectors $e_1, \cdots, e_n$, are linearly independent.)

By this theorem, three eigenvalues of A^{-3} are distinct
$\Rightarrow$ the eigenvectors are linearly independent.

(b)

```
In[43]:= λ2 = N[Eigenvalues[Inverse[A].Inverse[A].Inverse[A]]]
Out[43]= {-0.0862816, 0.0511949, -0.0115018}
In[44]:= N[Eigenvectors[Inverse[A].Inverse[A].Inverse[A]]]
Out[44]= {{-10.4047, 30.9511, 1.}, {1.63923, -0.224562, 1.}, {-0.234525, -1.72651, 1.}}
```

(c) Show that $B^5 x = \lambda^5 x$.

$Bx = \lambda x$

$B \cdot Bx = B \cdot \lambda x$

$(B)^2 x = \lambda Bx = (\lambda)^2 x$

$\cdots \quad (B)^5 x = (\lambda)^5 x$

$\therefore$ eigenvalues of B^5 : λ^5, where λ is eigenvalues of B.

eigenvectors of B^5 : X, where X is eigenvectors of B

```
In[45]:= B = (5  1  2
              0  4  3
              3 -2 -1)
Out[45]= {{5, 1, 2}, {0, 4, 3}, {3, -2, -1}}

In[46]:= λ3 = N[Eigenvalues[B]]
Out[46]= {6.04268, 2.31478, -0.357463}

In[47]:= λ3^5
Out[47]= {8056.55, 66.4582, -0.0058365}

In[48]:= N[Eigenvectors[B]]
Out[48]= {{3.32667, 1.46866, 1.}, {-0.0818619, -1.78018, 1.}, {-0.244804, -0.688474, 1.}}
```

By the theorem, because three eigenvalues of B^5 are distinct,

the eigenvectors are linearly independent.

(d)

```
In[49]:= λ4 = N[Eigenvalues[B.B.B.B.B]]
Out[49]= {8056.55, 66.4582, -0.0058365}

In[50]:= N[Eigenvectors[B.B.B.B.B]]
Out[50]= {{3.32667, 1.46866, 1.}, {-0.0818619, -1.78018, 1.}, {-0.244804, -0.688474, 1.}}
```

3\. (a) In order that P is singular,

$$\det P = 0 \quad \&$$

P^{-1} does not exist.

Assume that A^{-1} exists and therefore $\det(A) \neq 0$

$$A^p = 0. \quad — \text{①}$$

Multiplying ① by A^{-1}, $\quad A^{-1}A^p = 0$

$$\rightarrow A^{-1}AA^{p-1} = IA^{p-1} = 0.$$

$$\rightarrow A^{p-1} = 0$$

Multiply again by A^{-1}, $\quad A^{-1}A^{p-1} = 0$

$$\rightarrow A^{-1}AA^{p-2} = IA^{p-2} = 0$$

$$\rightarrow A^{p-2} = 0$$

contradiction!

$$\cdots \quad A = 0.$$

$A = 0$ means that $\det(A) = 0$.

$\therefore$ A^{-1} does not exist & $\det(A) = 0$

$\rightarrow$ A is singular //

(b) $(I-A)^{-1} = I + A + A^2 + \cdots A^{p-1}$.

Multiplying both sides by $(I-A)$

Left Hand Side $= (I-A)(I-A)^{-1} = I$

Right Hand Side $= (I-A)(I + A + A^2 + \cdots A^{p-1})$

$$= I + \cancel{A} + \cancel{A^2} + \cdots \cancel{A^{p-1}} - \cancel{A} - \cancel{A^2} - \cancel{A^3} - \cdots - A^p$$

$$= I - A^p$$

$$= I \quad (\because A^p = 0)$$

$\therefore$ LHS = RHS = I ,

4. (a) $$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$$

$$\det A = 1 \cdot \left(\frac{1}{15} - \frac{1}{16}\right) - \frac{1}{2} \cdot \left(\frac{1}{10} - \frac{1}{12}\right) + \frac{1}{3}\left(\frac{1}{8} - \frac{1}{9}\right)$$

$$= \frac{1}{2160}$$

$$\operatorname{adj} A = \begin{bmatrix} +\left(\frac{1}{15} - \frac{1}{16}\right) & -\left(\frac{1}{10} - \frac{1}{12}\right) & +\left(\frac{1}{8} - \frac{1}{9}\right) \\ -\left(\frac{1}{10} - \frac{1}{12}\right) & +\left(\frac{1}{5} - \frac{1}{9}\right) & -\left(\frac{1}{4} - \frac{1}{6}\right) \\ +\left(\frac{1}{8} - \frac{1}{9}\right) & -\left(\frac{1}{4} - \frac{1}{6}\right) & +\left(\frac{1}{3} - \frac{1}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{240} & -\frac{1}{60} & \frac{1}{72} \\ -\frac{1}{60} & \frac{4}{45} & -\frac{1}{12} \\ \frac{1}{72} & -\frac{1}{12} & \frac{1}{12} \end{bmatrix}$$

$$A^{-1} = 2160 \begin{bmatrix} \frac{1}{240} & -\frac{1}{60} & \frac{1}{72} \\ -\frac{1}{60} & \frac{4}{45} & -\frac{1}{12} \\ \frac{1}{72} & -\frac{1}{12} & \frac{1}{12} \end{bmatrix} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

$$X = A^{-1} C = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -12 \\ 30 \end{bmatrix}$$

$$X = [-3, \ -12, \ 30]^T$$

(b)

In[51]:= $A1 = \begin{pmatrix} 1 & 0.5 & 0.33 \\ 0.5 & 0.33 & 0.25 \\ 0.33 & 0.25 & 0.2 \end{pmatrix}$

$c = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Out[51]= {{1, 0.5, 0.33}, {0.5, 0.33, 0.25}, {0.33, 0.25, 0.2}}

Out[52]= {{1}, {2}, {2}}

In[55]:= **Inverse[A1].c // MatrixForm**

Out[55]//MatrixForm=

$\begin{pmatrix} 11.1111 \\ -84.127 \\ 96.8254 \end{pmatrix}$ //

(c) In[56]:= $A2 = \begin{pmatrix} 1 & 0.5 & 0.333 \\ 0.5 & 0.333 & 0.25 \\ 0.333 & 0.25 & 0.2 \end{pmatrix}$

Out[56]= {{1, 0.5, 0.333}, {0.5, 0.333, 0.25}, {0.333, 0.25, 0.2}}

In[57]:= **Inverse[A2].c // MatrixForm**

Out[57]//MatrixForm=

$\begin{pmatrix} -2.77854 \\ -13.0389 \\ 30.9249 \end{pmatrix}$ //

5.

$$Ae = \lambda e$$

Taking dot product on both sides

$$e \cdot (Ae) = e \cdot (\lambda e)$$

Ae is a column vector, so $e \cdot (Ae) = e^T Ae$

$$\therefore \quad e^T Ae = e \cdot (\lambda e)$$

Because λ is a scalar, $e \cdot (\lambda e) = \lambda (e \cdot e) = \lambda e^T e$

$$\therefore \quad e^T Ae = \lambda e^T e$$

$$\lambda = \frac{e^T Ae}{e^T e} \quad //$$

6. (a)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det [A - \lambda I] = 0$ is the characteristic equation.

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0.$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

By the Cayley – Hamilton theorem,

$$A^2 - (a+d)A + (ad-bc)I = 0$$

$\Rightarrow$ prove this.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+dc & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{a^2}+\cancel{bc}-\cancel{a^2}-\cancel{ad}+\cancel{ad}-\cancel{bc} & \cancel{ab}+\cancel{bd}-\cancel{ab}-\cancel{bd} \\ \cancel{ac}+\cancel{dc}-\cancel{ac}-\cancel{dc} & \cancel{bc}+\cancel{d^2}-\cancel{ad}-\cancel{d^2}+\cancel{ad}-\cancel{bc} \end{pmatrix}$$

$$= 0 \quad //$$

(b) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$A^2 - 4A + 3I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 0$$

$$\therefore A^2 - 4A + 3I = 0. \quad — ①$$

Multiply A^{-1} on both sides of ①

$$A^{-1}AA - 4A^{-1}A + 3A^{-1}I = 0$$

$$IA - 4I + 3A^{-1} = 0$$

$$A^{-1} = \frac{4}{3}I - \frac{1}{3}A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad //$$