**Introduction to Continuum Mechanics**

In this course, we will treat fluids like continua; in other words, we are going to ignore the molecular granularity of matter. This is an assumption which we, as engineers, often make in describing transport of heat, mass, or momentum although we don’t always state this assumption explicitly. To make the nature of this assumption clearer, it might help to discuss the alternative.

Fluids are composed of molecules. In principle, if you tell me the initial location of every molecule in the fluid and its initial velocity, I can compute the position and velocity at some later time by applying Newton’s laws of motion (i.e. \( F = ma \)) to each molecule during each collision with another molecule. The difficulty with this approach is that the number of molecules in any volume of fluid of interest to us make such a detailed calculation impractical. For example:

1 cm³ of water \( \rightarrow 3.3 \times 10^{22} \) molecules
\( \rightarrow 1 \) million years

Even with a computer operating at 1000 mflops, it would take 1 million years to do just one multiplication for each molecule. Molecules of a liquid collide on the average of once every \( 10^{12} \) seconds. To describe one second of real behavior, I would need \( 10^{12} \) million years. Clearly, this is an absurd length of time. Although computers get faster every year, this will remain an absurdly long time for the foreseeable future. The alternative is:

**Continuum Hypothesis**

A detailed description at the molecular level is not required in order to predict macroscopic behavior of any material. For example, it is not necessary to know the precise location of every molecule of fluid; it turns out that all that is needed for most applications is the distribution of mass described by the density profile \( \rho(r) \) of molecules in some region:

\[
\rho(r) = \lim_{V \to 0} \left\{ \frac{1}{V} \sum_{i} m_i \right\}
\]

where \( m_i \) is the mass of molecule \( i \), the sum is over all the molecules inside surface \( A \), and \( V \) is the volume. In fluid mechanics, as in heat and mass transfer, we make an assumption known as the "continuum hypothesis."

![Graph](image)

Basically this assumption is that the limit above will converge long before the dimensions of \( V \) shrink to molecular size. Similarly, we don't need to know the translational, rotational, vibrational and electronic energy of each molecule. We usually need only to know the internal energy per unit volume as a function of position, which in turn, manifests itself macroscopically as temperature.

A more precise statement of the continuum hypothesis is:

**Continuum Hypothesis** - the region to be described can be subdivided into a set of (infinitesimal) volume elements, each of which simultaneously:

1. is small enough to be considered uniform (i.e. any spatial variations in properties -- such as \( \rho, v, T, p \) -- inside the volume element are negligible); and

2. is large enough to contain a statistically large number of molecules.

In other words, we are assuming that \( dV \) exists such that the two conditions above are both satisfied. Materials which obey this “hypothesis” as said to behave as a continuum. Generally, the continuum hypothesis works well provided all the dimensions of the system are large compared to molecular size. An example of a situation in which the continuum hypothesis does not work is the flow of dilute gases in small pores, where the mean free path (for the collision of molecules) is comparable to the dimensions of the pore. This situation is called “Knudsen diffusion.”

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The basic problem in continuum mechanics is to describe the response of material to stress. A quantitative statement of that response is known as:

**Constitutive Equation** - model which describes how a material will respond to stress.

Familiar examples of constitutive equations include:

- Hooke's law of elasticity (solids)
- Newton's law of viscosity (fluids)

Many materials, like toothpaste and polymer melts, have characteristics of both solids and fluids and do not obey either of these simple "laws." Such fluids are called **viscoelastic**.

**Classification of Forces**

Having derived an equation (the Continuity equation) to describe the relationship among the variables which is imposed by conservation of mass, the remaining fundamental principle of physics is Newton’s second law ($\sum F = ma$) which, as it turns out, is equivalent to conservation of momentum. To apply this principle, we will need to list the forces which can act on fluid systems. Forces tend to fall into one of two different categories, depending on the range over which they act: long-range (compared to molecular size) forces are computed as volume integrals (called “body forces”) and short-range forces are computed as surface integrals (called “surface forces”).

Of course, gravitational forces have the longest range of any known force. For example, gravitational forces between planets and the sun determine their orbits. In particular, all fluid elements (not just those at the system boundary) feel a gravitational force of interaction with the rest of the universe outside the system boundaries. Thus gravity is a “body force.”

**body forces**: those which act on every fluid element in body (e.g. gravity):

$$dF_g = (dm)\mathbf{g} = \rho \mathbf{g} \, dV$$

where $\mathbf{g}$ is the acceleration of gravity (a vector pointing toward the center of the earth).

At the other end of the spectrum are forces which have very short range. If the range is of molecular dimensions, then only fluid elements experience a nonzero interaction with the universe outside the system. Although interior fluid elements might interact with one another through this short-range force, this interaction is not considered in a force balance, because the “action” and “reaction” forces cancel, leaving no net contribution to the force on the system. When only surface elements feel a particular force from outside, that force is called a “surface force.”

At the molecular scale, pressure arises from the momentum transferred during collisions between molecules outside and molecules inside the system. Since only surface molecules will be struck from outside, pressure is a surface force.

**surface forces**: those which act only on surfaces (including mathematical boundaries)

One example is **hydrostatic pressure**.
\[ dF_p = -p_n \, da \]

where \( dF_p \) is the force exerted on the system (through the surface element \( da \)) by the fluid outside, and \( n \) is a unit outward (to system) normal. For a proof that this is the correct form for hydrostatic pressure, see Batchelor, Section 1.3.

**Hydrostatic Equilibrium**

If our material is a fluid and if it is at rest (no velocity and no acceleration), then gravity and hydrostatic pressure forces are usually the only forces acting on the system. At equilibrium, the forces must be balanced. Thus Newton's 2nd law, which generally requires

\[ \sum_i F_i = Ma, \]

reduces to

\[ \sum_i F_i = 0 \]

at mechanical equilibrium. In our case, this means

\[
\begin{align*}
F_g + F_p &= 0 \\
F_g &= \int_V p g \, dV \\
F_p &= \int_A p_n \, da = -\int_V \nabla p \, dV
\end{align*}
\]

To obtain this last result, we applied one of the corollaries of the Divergence Theorem. Substituting back into the force balance and combining the two volume integrals leads to:

\[
F_g + F_p = \int_V (p g - \nabla p) \, dV = 0
\]

Since \( V \) is arbitrary, we conclude that the integrand vanishes:

\[ \nabla p = \rho g \]

This says that the pressure increases in the direction of the acceleration of gravity (downward), which correctly describes (for example) how the pressure increases with depth in an ocean.

**Flow of Ideal Fluids**

Now let's consider fluids in motion. The simplest analysis is for:

*ideal fluid* - deformation of fluid elements is an isentropic process (i.e. adiabatic and reversible):

\[ \mu = 0 \text{ and } k = 0 \]

where \( \mu \) is the viscosity and \( k \) is the thermal conductivity. Generally this means that any viscous forces are negligible (since viscous forces represent friction arising between fluid elements and friction gives rise to irreversibility). Furthermore, to keep the process adiabatic, the thermal conductivity must also be negligible.

**Euler's Equation**

Suppose these conditions on the fluid are met. Thus consider the isentropic deformation of an ideal fluid for an arbitrary macroscopic system. In addition to pressure and gravity, we must also consider inertia when the system accelerates. Newton's law requires:

\[ Ma = \Sigma_i F_i \]

Let \( \mathbf{r}(t) \) denote the trajectory of one particular fluid element inside the system. Then the velocity of the fluid element is:

\[ \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} \]

---

* Sir Isaac Newton (1642–1727), English mathematician and natural philosopher (physicist); considered by many the greatest scientist of all time; invented differential calculus and formulated the theories of universal gravitation, terrestrial mechanics, and color.
\[ v = \frac{Dr}{Dt} \]

while the acceleration is:

\[ a = \frac{Dv}{Dt} \]

We use the material derivative here, since we are following a particular material point. Multiplying the acceleration by the mass of the fluid element gives the inertia:

\[ (dm)a = \rho \left( \frac{dV}{dt} \right) \frac{Dv}{Dt} \]

To get the net inertia of the entire system, we must repeat this calculation for each of the fluid elements composing the system and add them up:

\[ \langle Ma \rangle = \int \rho \frac{Dv}{Dt} dV \]

I’ve put cares (\( \langle \ldots \rangle \)) around the \( Ma \) to indicate that this is the position-average inertia of the system (since the local \( (dm)a \) varies from point to point within a general system). Newton’s second law requires us to equate this with the net force acting on the system:

\[ \int_V \rho \frac{Dv}{Dt} dV = F_p + F_g = \int_V \rho g \, dV - \int_A n_p \, da \]

Using the divergence theorem to convert the surface integral into a volume integral, we have three volume integrals over the same domain. Combining these three volume integrals leaves:

\[ \int_V \left[ \rho \frac{Dv}{Dt} - \rho g + \nabla \rho \right] dV = 0 \]

Since this must hold for any choice of \( V \), the integrand must vanish at each point in the domain. After dividing by \( \rho \):

\[ \frac{Dv}{Dt} = g - \frac{1}{\rho} \nabla \rho \]  

which is called Euler’s Equation (1755).*

**Significance:** When combined with a statement of continuity, Euler’s equation of motion provides as many equations as unknowns.

Another relationship among the unknowns is the continuity equation (see page 17), which comes from the mass balance.

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \]

For an incompressible fluid, \( \rho = \text{const.} \) w.r.t. both time and position. Then the continuity equation reduces to:

\[ \nabla \cdot v = 0, \quad (2) \]

To see that we now have as many equations as unknowns, note that the unknowns in (1) and (2) are unknowns: 4 scalars \( v \) and \( p \)

which represents the 3 scalar components of \( v \) plus \( p \), for a total of 4 scalar unknowns (\( \rho \) and \( g \) are considered to be known). To evaluate these unknowns, we have equations (1) and (2):

equations: 4 scalars Euler + continuity

but Euler’s equation (1) is a vector equation, which can be expanded into 3 independent scalar equations. When added to continuity (a scalar equation), we obtain a total of 4 independent scalar equations, the same number as of scalar unknowns. Thus we are now in position to begin solving problems involving fluid flow. We will call (1) and (2) “Euler’s equations of motion for incompressible fluids.”

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* Euler, Leonhard, 1707-83, Swiss mathematician. The most prolific mathematician who ever lived, he worked at the St. Petersburg Academy of Sciences, Russia (1727-41, 1766-83), and at the Berlin Academy (1741-66). He contributed to areas of both pure and applied mathematics, including calculus, analysis, number theory, topology, algebra, geometry, trigonometry, analytical mechanics, hydrodynamics, and the theory of the moon’s motion.

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EXAMPLE. Water in a partially filled tank undergoes uniform\* acceleration \( \mathbf{a} \) in the horizontal plane. Find the angle \( \theta \) of inclination of the water’s surface with respect to the horizontal plane.

Solution. The key to solving this problem is to recognize that, regardless of the angle of inclination, the pressure is equal to 1 atm everywhere on the free surface. Recall from the geometric meaning of gradient (see page 3) of a scalar that the gradient of any scalar field (e.g. \( p \)) is a vector \( \nabla p \) which is normal to a \( p=\text{const} \) surface. Since the air/water interface is the \( p=1 \) atm surface, the pressure gradient \( \nabla p \) must be normal to this plane. If we can find the orientation of \( \nabla p \) we will have the orientation of the free surface. Recall Euler's equation of motion for an ideal fluid:

\[
\frac{D \mathbf{v}}{Dt} = \mathbf{g} - \frac{1}{\rho} \nabla p \nabla p
\]

\( Dv/\!\!Dt \) is just the acceleration of the fluid in a stationary reference frame. At steady state, all of the fluid will undergo the same uniform acceleration as the tank; so \( Dv/\!\!Dt \) is just \( \mathbf{a} \). Solving for the gradient, we have

\[
\frac{1}{\rho} \nabla p = \mathbf{g} - \mathbf{a}
\]

Using vector addition in the drawing at right, we can see that the angle of inclination of the free surface (relative to the horizon) is just

\[
\theta = \tan^{-1} \left( \frac{a}{g} \right)
\]

We were lucky in the previous example, because we knew the left-hand side of (1), so instead of 4 scalar unknowns, we only had one: \( p \). The solution was relatively easy. In the more general problem, the left-hand side of (1) is an unknown nonlinear partial differential equation:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{g} - \frac{1}{\rho} \nabla p \tag{3}
\]

In this form, we have expanded \( Dv/\!\!Dt \) using the relationship between material derivative and partial derivative (see page 10). Now we have 4 scalar unknowns: the three scalar components of \( \mathbf{v} \) and pressure: \( v_x, v_y, v_z \) and \( p \). Coupled with the continuity equation (for an incompressible fluid)

\[
\nabla \cdot \mathbf{v} = 0
\]

Euler’s equation also gives us 4 scalar equations. One important class of solutions has the form \( \mathbf{v} = \nabla \phi \), which is called “potential flow.” In the next section, we discuss how this form comes about and identify which physical problems have this form.

**Kelvin’s Theorem**

An important precursor to the theory of potential flow is the principle of conservation of circulation. Before stating this principle, let me define a quantity

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\* By “uniform acceleration” I mean the same acceleration is experienced at each point and at each time. For a fluid, uniformity at each position occurs only in the steady state after a transient which is nonuniform.
which Landau & Lifshitz call the velocity circulation:

\[ \Gamma = \oint_C \mathbf{v} \cdot d\mathbf{r} \]

for any closed contour. Recall that we showed on page 19 that this contour integral is associated with the average angular momentum of fluid elements located on the surface whose edge is \( C \). Kelvin showed that this velocity circulation is conserved:

\[ \frac{d\Gamma}{dt} = 0 \]

for any set of material points forming a closed contour in an ideal fluid. This result is called Kelvin’s Theorem.

\[ \text{contours
of material points
at different times} \]

\[ \text{trajectory of a
material point} \]

\[ \text{a material point} \]

Partial proof\( ^\star \) Since the contour \( C \) is composed of material points, the time derivative of this contour integral is like the material derivative:

\[ \frac{d\Gamma}{dt} = \frac{D}{Dt} \oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_C \frac{D}{Dt} \mathbf{v} \cdot d\mathbf{r} \]

Since we are always integrating over the same set of material points, a boundary term does not arise when we interchange integration and differentiation operators, although the set of spatial points is time-dependent: \( C = C(t) \). Of course, we have not rigorously shown this step to be valid, thus we only claim the proof is “partial.” Next, we substitute Euler’s equation and write each term as the gradient:

\[ \frac{d\Gamma}{dt} = \oint_C \left( \mathbf{g} - \frac{1}{\rho} \nabla p \right) \cdot d\mathbf{r} = \oint_C \left( \nabla \phi_g - \nabla \frac{P}{\rho} \right) \cdot d\mathbf{r} = 0 \]

In the second equality above, we have introduced the potential energy per unit mass, \( \phi_g \). Recall that gravity is a conservative force field (see page 20). For an object of constant mass (e.g., a brick), Theorem III guarantees that a scalar field \( \phi(\mathbf{r}) \) exists such that \( \mathbf{F}_g = mg = -\nabla \phi \). For an object of constant mass, we could divide both sides of this equation by \( m \) and bring \( m \) inside the gradient operator: \( \mathbf{g} = -\nabla(\phi/m) \).

Similarly, a differential volume element \( dV \) would have a differential mass \( dm \) and a differential potential energy \( d\phi \), such that \( \mathbf{g} = -\nabla(d\phi/dm) \). So generally we can write:

\[ \mathbf{g} = -\nabla \phi_g \]  

(4)

where

\[ \phi_g = \frac{\text{potential energy}}{\text{mass}} \]

In the second equality above, we have also brought \( \rho \) inside the gradient operator:

\[ \frac{1}{\rho} \nabla p = \nabla \frac{P}{\rho} \]  

(5)

This is only correct when \( \rho \) can be treated as spatially constant (e.g., an incompressible fluid). The next-to-last expression (in the equation for \( d\Gamma/dt \)) must vanish, because it has the form \( \nabla \mathbf{s} \cdot d\mathbf{r} = ds \); integrating any total differential around a closed contour yields zero (Theorem III).

\[ \oint_C \mathbf{v} \cdot d\mathbf{r} = \text{const. w.r.t. time} \]

Keep in mind that this applies only if \( C \) is composed of material points and only for ideal flow. Since \( C \) is composed of material points (which in general move with different velocities), the contour may change.


\( ^\dagger \) Lord Kelvin (William Thomson), 1st Baron, 1824-1907, British mathematician and physicist; b. Ireland. He was professor (1846-99) of natural philosophy at the Univ. of Glasgow. His work in thermodynamics coordinating the various existing theories of heat established the law of the conservation of energy as proposed by James Joule. He discovered what is now called the “Thomson effect” in thermoelectricity and introduced the Kelvin scale, or absolute scale, of temperature. His work on the transmission of messages by undersea cables made him a leading authority in this field.

\( ^\blacklozenge \) For a more rigorous proof, see Batchelor p269. For a more intuitive proof, see L&L, p15.
shape or move. Given the meaning of this contour integral (\(\Gamma = \) angular momentum), this result implies that (in the absence of friction) angular momentum of fluid is a constant. In general, we change the angular momentum of some object by applying a torque. So this result (i.e. Kelvin’s theorem) means that (in the absence of friction) there is no way to apply a torque to ideal fluid elements.

If you think about it, this makes sense: the usual way to apply a torque (with our hands to a cylinder, say) is to hold the cylinder between our hands and then move our hands in opposite directions, as shown in the sketch at right. We thus rely on friction between our hands and the cylinder to exert the torque. If the cylinder were greased and our hands slipped over its surface, we would not be able to apply the torque. This is the essence of what Kelvin’s theorem is saying.

\textbf{Irrotational Flow of an Incompressible Fluid}

As an example, consider towing a submerged object through an “ideal fluid” which is otherwise stagnant.

\[
v(r, t=0) = 0
\]

Consider an arbitrary closed contour in the fluid far from the disturbance caused by the motion of the submerged object. The contour integral vanishes since \(v\) vanishes:

\[
\oint_C v \cdot dr = 0 \quad \text{for} \quad t = 0 \quad \text{since} \quad v = 0
\]

for every such contour \(C\). Now at some later time the submerged object moves into the vicinity of \(C\) which causes \(v\) to be nonzero. Despite this, Kelvin’s Theorem still requires

\[
\oint_C v \cdot dr = 0 \quad \text{for all} \quad t \quad \text{although} \quad v \neq 0
\]

This is also true for every closed contour in the region (since every contour initially had zero value for this integral).

Applying Theorem III:

\[
\nabla \times v = 0 \quad \text{for all} \quad r, t
\]

which is sometimes called the \textit{persistence of irrotationality}. Also from Theorem III, we know \(\phi(r, t)\) exists such that:

\[
v = \nabla \phi
\]

where \(\phi\) is called the velocity potential.

\textbf{Significance:} Knowing that the solution has the form given by (7) allows us to decouple the four scalar equations represented by (1) and (2):

\[
\nabla \cdot \nabla \phi = 0
\]

Substituting (7) into (2) yields Laplace’s equation in the velocity potential:

\[
\nabla \cdot \nabla \phi = 0
\]

Instead of 4 equations in 4 unknowns, we now have a single equation which can be solved for \(\phi\) and \(v = \nabla \phi\), without any coupling to Euler’s equation (1). Although Euler was the first to suggest this approach, this is called \textit{Laplace’s Equation} after the French mathematician who solved this equation in so many cases.

Knowing the velocity profile \(v\), we can now determine the pressure profile \(p\) from Euler’s equation:

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v = g - \frac{1}{\rho} \nabla p
\]

We will now integrate this vector equation to obtain a single scalar equation for the pressure profile. Each term in (8) can be expressed as a gradient of something. For example, we’ve already seen in (4) that:

\[
g = -\nabla \phi_g
\]

Similary, from (5):

\[
\Phi = \Phi_g
\]

\[
\Phi_g = \frac{1}{2} \rho v^2 + \frac{1}{2} \rho \Omega^2 r^2 + 
\]

\[
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\]

* Pierre Simon de Laplace (1749-1827), French mathematician and astronomer, noted for his theory of a nebular origin of the solar system and his investigations into gravity and the stability of planetary motion.
\[
\frac{1}{\rho} \frac{\partial p}{\partial t} = \nabla \left( \frac{p}{\rho} \right)
\]

For potential flow, the unsteady term becomes

\[
\frac{\partial \mathbf{v}}{\partial t} = \nabla \left( \frac{\partial \phi}{\partial t} \right) = \nabla \left( \frac{\partial \phi}{\partial t} \right)
\]

Finally, for the convective term, we can apply identity A.3:

\[
\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} \nabla \left( \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times \nabla \times \mathbf{v}
\]

where \( \mathbf{v} \cdot \mathbf{v} = v^2 \)

where \( v = |\mathbf{v}| \). In our particular case, the second term vanishes because \( \nabla \times \mathbf{v} = \mathbf{0} \) for potential flow. Substituting (4), (5), (9) and (10) into (8):

\[
\nabla \left( \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \phi_g + \frac{p}{\rho} \right) = \mathbf{0}
\]

which is called Bernoulli’s Equation. It implies that

\[
\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \phi_g + \frac{p}{\rho} = \text{const w.r.t. } \mathbf{r}
\]

is spatially uniform, but it might depend on time. Once the velocity profile is obtained (by solving Laplace’s equation), both \( \phi \) and \( v \) are known, leaving \( p \) as the only unknown.

Potential Flow Around a Sphere

To solve a typical problem involving potential flow, we would first solve Laplace's equation to obtain the velocity profile and then we can evaluate the pressure profile using Bernoulli’s equation. Let's illustrate the procedure using an example:

**EXAMPLE:** Find the velocity and pressure profiles for potential flow caused by a sphere of radius \( R \) moving through a stagnant fluid with velocity \( \mathbf{U} \).

**Solution:** If the fluid behaves ideally, it undergoes potential flow and the velocity profile must satisfy Laplace’s equation:

\[
\nabla^2 \phi = 0
\]

Boundary conditions can be formulated by recognizing that fluid far from the sphere is unperturbed:

b.c. #1: \( \mathbf{v} = 0 \) far from sphere

while fluid near the sphere cannot penetrate the sphere. To express this mathematically, recall our “bucket-and-stopwatch” method for defining fluid velocity (see page 9). Modifying it slightly to account for the movement of the surface element at velocity \( \mathbf{U} \), the flowrate across a surface element of area \( da \) is given by:

\[
dq = \mathbf{n} \cdot (\mathbf{v} - \mathbf{U}) \, da = 0
\]

For an impenetrable sphere, the flowrate must vanish

b.c. #2: \( \mathbf{n} \cdot (\mathbf{v} - \mathbf{U}) = 0 \) on sphere

Let reference frame move with object:

Owing to the moving boundary, the solution to this problem is inherently unsteady; i.e. \( \phi = \phi(\mathbf{r}, t) \). To avoid the unsteadiness, we adopt a new reference frame in which the origin moves along with the center of the sphere. It turns out that the PDE does not change upon this shift in reference frame for velocity: the new velocity potential must also satisfy Laplace’s equation:
P.D.E.: \[ \nabla^2 \phi = 0 \]

However, the boundary conditions are changed. In this moving coordinate system, the sphere appears to be stationary and the fluid at infinity is undergoing uniform flow.

\textbf{b.c. \#1:} \[ \mathbf{v} \rightarrow - \mathbf{U} = \mathbf{U} \mathbf{k} \text{ as } r \rightarrow \infty \quad (12) \]

where \( \mathbf{U} \) is the velocity of the sphere in the original stationary reference frame.

For a stationary sphere, no fluid entering the sphere means \( dq = \mathbf{n} \cdot \mathbf{v} \, da = 0 \), or

\textbf{b.c. \#2:} \[ \mathbf{n} \cdot \mathbf{v} = v_r = 0 \text{ at } r = R \]

Now the potential has a steady state solution. Next we rewrite the b.c.’s in terms of the velocity potential in spherical coordinates. In terms of velocity potential, (12) becomes:

\[ \nabla \phi = \mathbf{U} \mathbf{k} \]

Now we need to translate \( \mathbf{k} \) into the unit vectors in spherical coordinates.*

Referring to the figure above (see page 5), we note that \( \mathbf{e}_r, \mathbf{e}_\theta, \) and \( \mathbf{e}_z = \mathbf{k} \) all lie in the same \( \Phi=\text{const} \) plane (shaded region of left-hand figure above). If we shift all three unit vectors to the origin (recall that the origin is not part of the definition of any vector) and re-orient the \( \Phi=\text{const} \) plane to coincide with the plane of the page, then we get the figure above at right, from trigonometry of the right triangle it is apparent that

\[ \mathbf{k} = (\cos \theta) \mathbf{e}_r - (\sin \theta) \mathbf{e}_\theta \]

Thus the b.c. can be written as

as \( r \rightarrow \infty: \quad \nabla \phi \rightarrow \mathbf{U} \cos \theta \mathbf{e}_r - \frac{U \sin \theta}{r} \mathbf{e}_\theta \]

Equating corresponding components:

\[ \frac{\partial \phi}{\partial r} = U \cos \theta \quad \text{and} \quad \frac{1}{r} \frac{\partial \phi}{\partial \theta} = U \sin \theta \]

Integrating either PDE leads to

\textbf{b.c. \#1:} \[ \phi \rightarrow U \cos \theta + \text{const} \quad \text{as} \quad r \rightarrow \infty \]

where we have arbitrarily selected a value of zero for this “const”.* In order to translate b.c. \#2, use tables (see spherical coor\textsuperscript{d}ds\*) to express the gradient in spherical coordinates:

\[ \mathbf{v} = \nabla \phi \]

\[ v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi = \frac{\partial \phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \Phi} \mathbf{e}_\phi \]

* Like most energies in thermodynamics, the reference state for potential is arbitrary and can be chosen solely for convenience.

* http://www.andrew.cmu.edu/course/06-703/Vops_sph .pdf

* * *

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Dotting both sides by the unit vector \( \mathbf{n} = e_r \), then using b.c. #2:

\[
\nu_r = \frac{\partial \phi}{\partial r} = 0 \text{ at } r=R
\]

satisfies the P.D.E. and the b.c. at \( r=R \). Unfortunately, it does not satisfy the b.c. at \( r \to \infty \), so we will have to try another guess. Since the failure occurred with the b.c. at \( r \to \infty \), we look there for the form of our second guess:

\[
\phi = A r \cos \theta
\]

This too satisfies the P.D.E. and the b.c. at \( r \to \infty \) (provided \( A=U \)) but to satisfy the b.c. at \( r=R, A \) must be chosen as 0. The tells us that \( A \) should have different values at different \( r \)'s. So we try a third guess which is slightly more general than the second:

\[
\phi(r, \theta) = f(r) \cos \theta
\]

Substituting (16) into (13)-(15), we find that \( \cos \theta \) cancels out, leaving:

\[
r^2 f'' + 2 r f' - 2 f = 0
\]

\[
f' = 0 \text{ at } r=R
\]

\[
f = U r \text{ as } r \to \infty
\]

Thus we have reduced the problem of solving the P.D.E. to one of solving an O.D.E. We recognize this O.D.E. to be a Cauchy-Euler equation, which always has at least one solution of the form \( f = r^p \). The general solution turns out to be:

\[
f(r) = A r^2 + Br
\]

Using the b.c.'s, we can evaluate \( A \) and \( B \). The particular solution to this problem is:

\[\text{The general form of an } N^{\text{th}}\text{-order Cauchy-Euler equation is}
\]

\[
a_N x^N \frac{d^N y}{dx^N} + a_{N-1} x^{N-1} \frac{d^{N-1} y}{dx^{N-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = 0
\]

At least one of the \( N \) linearly independent solutions has the form \( y(x) = A x^\alpha \). Substituting this form and dividing out the common factor \( A x^\alpha \), we obtain an \( N^{\text{th}}\)-order polynomial for \( \alpha \):

\[
a_0 + a_1 \alpha + a_2 \alpha (\alpha - 1) + \ldots + a_N \alpha (\alpha - 1)(\alpha - 2)\ldots(\alpha - N + 1) = 0
\]

Each distinct root of the polynomial leads to a separate solution. In this example, the roots are \( \alpha = -1 \) and +1.
\[ f(r) = U \left( r + \frac{1}{2} \frac{R^3}{r^2} \right) \]

Substituting this back into (16) leaves:

\[ \phi(r, \theta) = U \left( r + \frac{1}{2} \frac{R^3}{r^2} \right) \cos \theta \]

\[ v_r = \frac{\partial \phi}{\partial r} = U \left[ 1 - \left( \frac{R}{r} \right)^3 \right] \cos \theta \]

\[ v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \left( 1 + \frac{1}{2} \frac{R^3}{r^3} \right) \sin \theta \]

Below are several attempts to display this solution graphically:

Recall that \( \theta \) is defined as follows:

Notice that for \( \theta = \pi \) (\( \theta = 0 \)), \( v_\theta = 0 \) but \( v_r \) decreases (in absolute magnitude) from \( U \) at \( r = \infty \) to 0 at \( r = R \). At \( \theta = \pi/2 \), \( v_r = 0 \) but \( v_\theta \) increases from \( U \) to \( (3/2)U \). This increase is necessary to make up for the decrease in flow caused by the sphere blocking part of the flow path.

Above is a plot of the trajectories of materials points around the sphere. These are called streamlines.

Having solved for the velocity profile, we can determine the pressure profile from Bernoulli’s equation (11). Assuming steady state:

\[ \frac{v^2}{2} + \frac{p}{\rho} + \phi_g = \text{const} \]

independent of position. The "const" can be evaluated using any point where we know both the velocity and the pressure. Suppose the pressure of the undisturbed fluid is known in the reference plane for the gravitational potential\( \nabla \):

at \( r \to \infty \), \( \phi_g = 0 \):

\[ p = p_\infty \text{ and } v^2 = U^2 \]

Thus

\[ \frac{v^2}{2} + \frac{p}{\rho} + \phi_g = \frac{U^2}{2} + \frac{p_\infty}{\rho} \]

or

\[ p(r, \theta) = p_\infty - \rho \phi_g + \frac{1}{2} \rho \left( U^2 - v^2 \right) \]

\( \nabla \) This is any horizontal plane in which we choose to define \( \phi_g = 0 \). It might pass through the center of the sphere or it might be the air-water interface at some distance above the sphere.
where \( v^2 = \mathbf{v} \cdot \mathbf{v} = \left[ v_r (r, \theta) \right]^2 + \left[ v_\theta (r, \theta) \right]^2 \)

Substituting the known velocity profile, we obtain the pressure profile. Let’s focus on the pressure profile over the surface of the sphere:

for \( r=R \):

\[
p(R, \theta, \Phi) = \frac{p_\infty - p_{\Phi}}{\text{hydrostatic head}} + \frac{1}{8} \rho U^2 \left( 9 \cos^2 \theta - 5 \right) \]

The hydrostatic pressure varies linearly with elevation (the x-axis is oriented vertically in our figure on page 31). The force arising from this contribution to pressure will be the subject of a problem on Hwk 4. We will here focus on the contribution from remainder. On the surface of the sphere, the dynamic pressure depends solely on \( \theta \).

In the sketch above, we plot the dynamic pressure (dropping the contribution from hydrostatic equilibrium). Note the location of regions having high and low pressure. The sphere is being pushed in at the poles and allowed to expand at the equator. This is why a large bubble rising through stagnant water tends to become distorted from spherical shape (see below). Such bubbles tend to become extended in the horizontal plane and compressed in the vertical direction by the higher pressure.

\[
d'\text{Alembert's Paradox}
\]

What is the net force on a rigid sphere owing the pressure profile developed by potential flow around it? The answer turns out to be:

\[
\mathbf{F}_p = -\oint \mathbf{n} \ p \ da = -\oint \mathbf{n} \ (p_h + p_d) \ da
\]

\[
= -\oint \mathbf{n} \ p_h \ da - \oint \mathbf{n} \ p_d \ da = -\rho g V \quad (17)
\]

where \( V \) is the volume of the sphere.

**Proof:** first consider the contribution from dynamic pressure:

\[
p_d = \frac{1}{8} \rho U^2 \left( 9 \cos^2 \theta - 5 \right)
\]

The \( \mathbf{n} \) in (17) is a unit vector normal to the surface, pointing outward. In spherical coordinates, with their origin at the center of the sphere, this vector is the unit vector in the \( r \)-direction:

\[
\mathbf{n} = \mathbf{e}_r(0, \theta)
\]

which direction depends on location on the surface. Although its length is a constant, its direction varies with position of the sphere; thus \( \mathbf{n} \) cannot be treated as a constant. Anticipating that any net force will be parallel to the direction of fluid flow, we dot both sides of (17) by \( \mathbf{k} \):

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\[ F_p = k \cdot F_p = - \int_A k \cdot \mathbf{n} \, p \, da \cos \theta \]

For the contribution from dynamic pressure, \( p_d(r=R, \theta) \) depends solely on \( \theta \), so we choose the strip of width \( R \, d\theta \) and length \( 2\pi R \, \sin \theta \) as our differential area \( da \). On this strip \( \theta \) is virtually constant.

\[
F_{dp} = -\int_0^\pi \cos \theta \, p_d(R, \theta) \left( 2\pi R \sin \theta \right) (R \, d\theta) \frac{d}{f(\cos \theta)}
= 2\pi R^2 \int_0^\pi \cos \theta \, f(\cos \theta) \left( -\sin \theta \, d\theta \right) \frac{d}{\cos \theta}
= 2\pi R^2 \times \frac{1}{8} \rho U^2 \int_0^1 x \left( 9x^2 - 5 \right) \, dx = 0
\]

The nonzero contribution from (17) comes from the hydrostatic head. We will leave this calculation as an exercise for the reader (HWK #4, Prob. 3). The net force due to pressure is

\[
\mathbf{F}_p = -\int_{sphere} \left( p_\infty - \rho \Phi_g \right) \mathbf{n} \, da
\]

\[
-\int_{sphere} \left[ \frac{1}{8} \rho U^2 \left( 9 \cos^2 \theta - 5 \right) \right] \mathbf{n} \, da = -\rho g V
\]

The net force on the sphere is the sum of its weight and the net pressure force:

\[
\mathbf{F}_p + \mathbf{F}_g = -\rho g V + \rho_s g V = (\rho_s - \rho) g V \neq 0
\]

which is the difference in weight of the sphere and the weight of the fluid displaced by it. Because the pressure force is independent of the speed of motion of the sphere through the fluid, the particle will continue to accelerate forever, without ever reaching a “terminal velocity.” Of course, experiments show that falling particles reach a terminal velocity which implies that the gravitational force is balanced by some other force. The other force is fluid drag (friction), which is not predicted by potential flow. This serious discrepancy between the predictions of potential flow theory and experiment is known as:

\[ \text{d’Alembert’s paradox} - \text{potential flow predicts no drag but experiments indicate drag.} \]

Despite this, potential flow is still useful:

\[ \text{Uses of potential flow} - \text{predicts lift (but not drag) on streamline objects moving through stagnant fluid at high Reynolds numbers (but still sub-sonic, i.e. } v \ll c). \]

- correctly predicts \( v(r,t) \) and \( p(r,t) \) except very near the surface of the object (i.e. inside boundary layer) and in wake [see pressure profile on airfoil shown below].

- for asymmetric shapes (e.g. airplane wings), it correctly predicts a lift.

The figure above shows pressure measurements (solid curves) taken along the top and bottom surfaces of an Zhukovskii aerfoil (a particular cross-sectional shape for the cylinder) and compares them to the potential flow predictions (dotted curves). These particular measurements were taken at a Reynolds number of \( 10^5 \) and an angle of attack \( (\alpha) \) of 6°.

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\[ \text{* Jean le Rond d’Alembert (1717–83), French mathematician and philosopher, a leading figure of the Enlightenment. His treatise on dynamics (1743) enunciated d’Alembert's principle, which permitted the reduction of a problem in dynamics to one in statics. He did important work on the mechanics of rigid bodies, the motions of fluids and vibrating strings, and the three-body problem in celestial mechanics.} \]

\[ \text{\textcopyright} \text{ Fig. 1.12 taken from Schlichting, 6th ed., p22f.} \]
Notice that the pressure is generally higher underneath the aerofoil than on top. So when pressure is integrated around the surface, an upward force is predicted.

This figure above shows how this upward lift force (expressed as a dimensionless lift coefficient) depends on the angle of attack. Measurements of the lift force (solid curve) are compared to predictions (dotted line). Except for high angles of attack, the two agree. For angles of attack above about 8°, the measured lift forces experience a maximum before decreasing. This decrease in lift with large angles is called “stall” and it arises because or large vortices formed in the wake of the aerofoil which are not predicted by potential flow theory.

* Fig. 1.13 taken from Schlichting, 6th ed.