06-155 Exam 2 **NAME:** KEY

Chemical Engineering Mathematics 06-155
EXAM 2
Tuesday March 9, 1999

A

Plan your time carefully:	Problem 1	20 points
	Problem 2	20 points
	Problem 3	20 points
	Problem 4	20 points
	Problem 5	20 points

1. Solve the following ordinary differential equations. Indicate the general solution and the particular solution for each. You do **not** need to check the solutions.

a) **(10 points)** $y' = x\,y\,\sin(x)$; $y(0) = 5$

$$\frac{dy}{dx} = x\sin(x)\cdot y \Rightarrow \int\frac{dy}{y} = \int x\sin(x)\,dx + c_1$$

$$\boxed{\ln y = -x\cos(x) + \sin(x) + c_1}$$ General

$y(0)=5$ $\quad \ln 5 = 0 + c_1 \Rightarrow c_1 = \ln 5$

$$\boxed{y(x) = \exp\{-x\cos(x) + \sin(x) + \ln 5\}}$$ Particular

b) **(10 points)** $y' = -(1 + x + y^2)/2y$; $y(0) = 3$

$$\frac{dy}{dx} = \frac{-(1+x+y^2)}{2y} \Rightarrow \underbrace{(1+x+y^2)}_{P(x,y)}dx + \underbrace{2y}_{Q(x,y)}dy = 0$$

Exact? $\frac{\partial P}{\partial y} = 2y \quad \frac{\partial Q}{\partial x} = 0$ No!

Look for integrating factor $F(x)$

$$\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} = \frac{1}{2y}\{2y - 0\} = 1$$

$$\int\frac{dF}{F} = \int dx \Rightarrow \ln F_1 = x \Rightarrow \boxed{F(x) = \exp(x)}$$

so $\underbrace{(1+x+y^2)e^x}_{M(x,y)}\,dx + \underbrace{2ye^x}_{N(x,y)}\,dy = 0$

since $\underbrace{\frac{\partial u}{\partial y}}= N$ and $\frac{\partial u}{\partial x} = M$

$$u(x,y) = \int N\,dy + k(x)$$
$$= \int 2ye^x\,dy + k(x) = y^2e^x + k(x)$$

$$\frac{\partial u}{\partial x} = M = (1+x+\cancel{y^2})e^x = \cancel{y^2e^x} + \frac{dk}{dx}$$

$$k(x) = (1+x)e^x - e^x$$

$$u(x,y) = C_1 = y^2e^x + (1+x)e^x - e^x$$
$$\boxed{(y^2+x)e^x = C_1} \quad \text{General}$$

$y(0)=3$ $\quad (9+0)e^0 = C_1 \rightarrow C_1 = 9$

$$\boxed{(y^2+x)e^x = 9} \quad \text{Particular}$$

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2. The mechanical behavior of polymeric materials is often described with a Maxwell Model. This model relates the stress (σ) generated in a material to the strain (ε) applied through a differential model. The Maxwell model is usually written:

$$\frac{d\varepsilon}{dt} = \frac{1}{G}\frac{d\sigma}{dt} + \frac{\sigma}{\eta}$$

where G = 10 Pa and η = 5 Pa•sec are material constants. You do **not** need to develop this ODE.

a) **(7 points)** In one case, we are applying a constant strain ($\varepsilon(t) = \varepsilon_o$). Develop an expression describing the stress as a function of time $\sigma(t)$ for this case.

b) **(3 points)** Using the expression developed in part (a) for the stress, use the condition that $\sigma(t{=}0) = G\varepsilon_o$ to evaluate the arbitrary constant.

c) **(7 points)** In a different case, a sinusoidal strain is applied ($\varepsilon(t) = \varepsilon_o\cos(\omega t)$). For this case, develop an expression describing the stress as a function of time $\sigma(t)$.

d) **(3 points)** Using the expression developed in part (c) for the stress, use the condition that $\sigma(t{=}0) = G\varepsilon_o\left(\frac{\omega^2}{(G/\eta)^2 + \omega^2}\right)$ to evaluate the arbitrary constant.

if $\varepsilon(t) = \varepsilon_o$ then $\frac{d\varepsilon}{dt} = 0$ so

$$\frac{1}{G}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = 0 \rightarrow \frac{d\sigma}{dt} = -\frac{G}{\eta}\sigma \quad \leftarrow \text{Separ}$$

$$\int \frac{d\sigma}{\sigma} = \int -\frac{G}{\eta}\,dt + c_1$$

$$\ln\sigma = -\frac{G}{\eta}t + c_1$$

$$\sigma(t) = c_1 \exp\left(-\frac{G}{\eta}t\right)$$

b) $\sigma(0) = G\varepsilon_o = c_1$

$$\sigma(t) = G\varepsilon_o \exp\left(-\frac{G}{\eta}t\right)$$

Problem 2 con't

c) $\varepsilon(t) = \varepsilon_0 \cos(\omega t) \longrightarrow \frac{d\varepsilon}{dt} = -\varepsilon_0 \omega \sin(\omega t)$

$$\frac{d\sigma}{dt} + \underbrace{\frac{G}{\eta}}_{p(t)}\sigma = \underbrace{-G\varepsilon_0\omega\sin(\omega t)}_{r(t)} \leftarrow \text{Linear FODE}$$

$$h = \int p(t)\,dt = \frac{G}{\eta}t$$

$$e^{\frac{G}{\eta}t}\sigma(t) = \int e^{\frac{G}{\eta}t}\left(-G\varepsilon_0\omega\sin(\omega t)\right)dt + C$$

$$= -G\varepsilon_0\omega\left\{\frac{e^{G/\eta\, t}}{\left(\frac{G}{\eta}\right)^2+\omega^2}\left(\frac{G}{\eta}\sin(\omega x) - \omega\cos(\omega x)\right)\right\} + C_1$$

$$\sigma(t) = \frac{G\varepsilon_0\omega}{\left(\frac{G}{\eta}\right)^2+\omega^2}\left(\omega\cos(\omega x) - \frac{G}{\eta}\sin(\omega x)\right) + C_1 e^{-\frac{G}{\eta}t}$$

$$\sigma(0) = G\varepsilon_0 \cancel{\frac{\omega^2}{(G/\eta)^2+\omega^2}} = \frac{G\varepsilon_0\omega}{\cancel{(G/\eta)^2+\omega^2}}(\omega) + C_1 \qquad C_1 = 0$$

a) $\cancel{Acc} = \text{In} - \text{Out} + \cancel{\text{Gen}} - \text{Cons}$

$$0 = \cancel{Qc} + q\,\cancel{\pi R^2} - \cancel{Qc} - \frac{d}{dx} Qc \cdot dx$$

$$- q\,\cancel{\pi R^2} - \frac{d}{dx}\, q \pi R^2 \cdot dx - kc\, \pi R^2 dx$$

$$0 = -\frac{d}{dx}(Qc)\cdot \cancel{dx} - \frac{d}{dx}(q\pi R^2)\cdot \cancel{dx} - kc\,\pi R^2 \cdot \cancel{dx}$$

use $q(x) = -D \frac{dc}{dx}$

$$0 = -Q \frac{dc}{dx} - \pi R^2 \frac{d}{dx}\left(-D\frac{dc}{dx}\right) - k \pi R^2 c$$

$$- Q \frac{\quad}{dx} -$$

$$\frac{d^2c}{dx^2} - \frac{Q}{\pi R^2 D} \frac{dc}{dx} - \frac{k}{D} c = 0$$

$$\frac{-Q}{\pi R^2 D} = -\frac{80\,\cancel{\pi}\,\cancel{L}}{\cancel{min}} \cdot \frac{1}{\cancel{\pi}(0.1\,m)^2} \cdot \frac{\cancel{min}}{16\,\cancel{m^2}} \cdot \frac{m^3}{10^3\,\cancel{L}}$$

$$a = -\frac{80}{10 \cdot 16} \frac{1}{m} = -\frac{1}{2}\, m^{-1} \quad //$$

$$b = -\frac{k}{D} = -\frac{1}{\cancel{min}} \cdot \frac{\cancel{min}}{16\, m^2} = -\frac{1}{16}\, m^{-2} \quad //$$

Problem 3 con't

b) $C'' \quad \frac{1}{2}c' - \frac{1}{16}c = 0$ x has units of m

$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{16} = 0$ $a = 1/2$, $b = -1/16$

$a^2 - 4b = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ ← 2 distinct real roots

$\lambda = \frac{1}{2}\left(+\frac{1}{2} \pm \sqrt{\frac{1}{2}}\right) = \frac{1}{4} \pm \frac{\sqrt{2}}{4} \Rightarrow \lambda_1 = \frac{1+\sqrt{2}}{4} = 0.604$

$\lambda_2 = \frac{-\sqrt{2}}{4} = -0.104$

$c(x) = c_1 e^{0.604x} + c_2 e^{-0.104x}$

c) $c(x=0) = c_0 = c_1 + c_2$

$c(x=L) = 0 = c_1 e^{\lambda_1 L} + c_2 e^{\lambda_2 L}$ $L = 3.28 \text{ ft} = 1 \text{ m}$

$\hookrightarrow -c_1 e^{\lambda_1 L} = c_2 e^{\lambda_2 L}$

$c_2 = -c_1 e^{(\lambda_1 - \lambda_2)L} = -c_1 e^{0.708L}$

$c_0 = c_1 + c_2 = c_1(1 - e^{0.708L}) = -1.03c_1$

$\therefore c_1 = -0.971\, c_0$

$c_2 = c_0 - c_1 = c_0(1 + 0.971) = 1.971c_0$

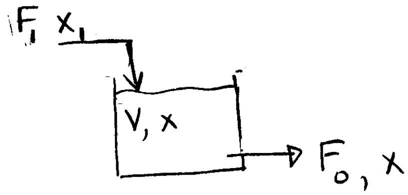
$$\boxed{C(x) = c_0\left\{-0.971\, e^{0.604x} + 1.971\, e^{-0.104x}\right\}}$$

4. Brine (salt + water solution) is fed into a mixing vessel at a rate of 3 ft^3/min and concentration of 2 lb/ft^3. The mixer is initially charged with 20 ft^3 of clean water. Liquid flows from the bottom of the tank at a rate of 2 ft^3/min. Assuming that the tank is well agitated and that the density is not dependent on salt concentration, what is the salt concentration in the tank when the tank contains 30 ft^3 of brine ?

a) **(5 points)** First, develop a differential equation that describes the rate of change of volume in the tank, and then determine a particular solution for V(t).

b) **(10 points)** Develop a mathematical model that describes the rate of change of the salt concentration in the tank. From this, develop a general and particular solution describing the salt concentration as a function of time.

c) **(5 points)** Using the relations derived in parts (a) and (b) determine the concentration of salt in the tank when the volume in the tank is 30 ft^3.



where x: conc of salt in lb/ft^3
V: volume in vessel
F_1, F_o : volumetric flow rates in ft^3/min

a) overall mass balance   conserved prop. = mass of liquid

$$\left.\begin{aligned} Acc &= \frac{d}{dt}(\rho V) \\ In &= \rho F_1 \\ Out &= \rho F_o \end{aligned}\right\} \qquad \rho\frac{dV}{dt} = \rho F_1 - \rho F_o \Rightarrow \frac{dV}{dt} = F_1 - F_o$$

$$V(t) = (F_1 - F_o)t + c \leftarrow \text{General}$$

Since $V(t=0) = 20 ft^3$

$$V(0) = c_1 = 20 ft^3$$

$$\boxed{V(t) = \left(3\frac{ft^3}{min} - 2\frac{ft^3}{min}\right)t + 20ft^3}$$

b) Salt balance   conserved prop. = mass of <u>SALT</u>

$$\left.\begin{aligned} Acc &= \frac{d}{dt}(Vx) \\ In &= F_1 x_1 \\ Out &= F_o x \end{aligned}\right\} \qquad \frac{d(Vx)}{dt} = F_1 x_1 - F_o x$$

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Problem 4 con't

but $\frac{d(Vx)}{dt} = V\frac{dx}{dt} + x\frac{dV}{dt}$ so

$$\frac{dx}{dt} = \frac{F_1 x_1 - F_0 x}{V} - \frac{x}{V}\underbrace{\frac{dV}{dt}}_{F_1 - F_0 \text{ from part (a)}}$$

note $x(t)$ and $V(t)$

$$\frac{dx}{dt} = \frac{F_1 x_1 - \cancel{F_0 x} - F_1 x + \cancel{F_0 x}}{V} = \frac{F_1(x_1 - x)}{V}$$

This is separable $\int \frac{dx}{(x_1 - x)} = \frac{F_1}{V}dt = \int \frac{F_1}{(F_1 - F_0)t + V_0}dt + C_2$

$$-\ln(x_1 - x) = F_1 \cdot \frac{1}{(F_1 - F_0)} \ln\{(F_1 - F_0)t + V_0\} + C_2$$

Remember $x_1 = 2\ lb/ft^3$ $\quad \frac{F_1}{F_1 - F_0} = 3 \quad F_1 - F_0 = 1\ ft^3/min$

$$\ln(2 - x) = -3\ln(t + V_0) + C_2$$

$$2 - x = \frac{C}{(t + V_0)^3} \Rightarrow \boxed{x(t) = 2 - \frac{C}{(t+V_0)^3}}$$

Since the tank is initially full of fresh water $x(t=0) = 0$

$$2 - \frac{C}{V_0^3} \Rightarrow C = 2V_0^3 \qquad \boxed{x(t) = 2\left\{1 - \left[\frac{V_0}{t+V_0}\right]^3\right\}}$$

The tank has a volume of $30\ ft^3$

when? $30\ ft^3 = (1\ ft^3/min)t + 20\ ft^3 \rightarrow t = 10\ min$

$$\boxed{x(10\ min) = 1.41\ \frac{lb}{ft^3}}$$ 7

5. The following second order differential equation arises when describing waves traveling along a rectangular channel with oscillating boundaries:

$$y'' - 4\,y' + 4\,y = -16 - 13\sin(3x) - 24\cos(2x)$$

a) **(6 points)** Determine the solution to the homogeneous part of the equation, $y_h(x)$
b) **(6 points)** Determine the solution to the nonhomogeneous part of the equation, $y_p(x)$
c) **(4 points)** Write the general solution of the differential equation.
d) **(4 points)** If the homogeneous part of the equation were $y'' + 9\,y = 0$, would your choice of $y_p(x)$ change ? If so, what would your choice be for $y_p(x)$?

a) $y_h'' \;\; 4y_h' + 4y_h = 0$

$\lambda^2 - 4\lambda + 4 = 0 \qquad a^2 - 4b = 16 - 16 = 0 \leftarrow$ double root

$$\boxed{y_h(x) = c_1 e^{2x} + c_2 x e^{2x}}$$

b) chose $y_p(x)$ based on $r(x) = -16 - 13\sin(3x) - 24\cos(2x)$

$y_p(x) = K_0 + A\sin(3x) + B\cos(3x) + M\sin(2x) + N\cos(2x)$

$y_p' = 3A\cos(3x) - 3B\sin(3x) + 2M\cos(2x) - 2N\sin(2x)$

$y_p'' = -9A\sin(3x) - 9B\cos(3x) - 4M\sin(2x) - 4N\cos(2x)$

plug back into $y_p'' - 4y_p' + 4y_p = r(x)$

and match terms...

Problem 5 con't

x^0 terms: $4K_o = -16 \longrightarrow K_o = -4$

$\sin(3x)$ terms: $-9A - 4(-3B) + 4(A) = -13$

$-5A + 12B = -13 \longrightarrow \frac{25}{12}B + 12B = -13$

$\frac{169}{12}B = -13$

$\cos(3x)$ terms: $-9B - 4(-3A) + 4(B) = 0$

$-12A - 5B = 0 \longrightarrow A = -\frac{5}{12}B$

$B = \frac{-12}{13}$

$A = \frac{5}{13}$

$\sin(2x)$ terms: $-\cancel{4M} - 4(-2N) + \cancel{4M} = 0$

$8N = 0 \longrightarrow N = 0$

$\cos(2x)$ terms: $-\cancel{4N} - 4(2M) + \cancel{4N} = -24$

$-8M = -24 \rightarrow M = 3$

$$\boxed{y_p(x) = -4 + \frac{1}{13}\left\{5\sin(3x) - 12\cos(3x)\right\} + 3\sin(2x)}$$

General solution is $\boxed{y(x) = y_h(x) + y_p(x)}$

d) If $y'' + 9y = 0$ were the homogeneous part, it would have the $y_h(x) = c_1\sin(3x) + C_2\cos(3x)$

So $y_p(x) = K_o + Ax\sin(3x) + Bx\cos(3x) + M\sin(2x) + N\cos(2x)$