Starred problems are for students enrolled in 80-713

1. (a) Fix a small category $\mathcal{C}$. Show that taking the category of elements of presheaves, together with the projection functor, determines a functor

$$\int_{\mathcal{C}} : \hat{\mathcal{C}} \rightarrow \text{Cat}/C,$$

taking $P \in \hat{\mathcal{C}}$ to $\pi : \int_{\mathcal{C}} P \rightarrow C$.

(b) Show that if $P$ is a poset and $A : P^{\mathrm{op}} \rightarrow \text{Sets}$ a presheaf on $P$, then the category of elements $\int_{P} A$ is also a poset and the projection $\pi : \int_{P} A \rightarrow P$ is a monotone map.

2. The $\mathcal{L}$ be a theory in the $\lambda$-calculus with $(1, \times, \rightarrow)$. For any type symbols $\sigma$ and $\tau$, let

$$[\sigma \rightarrow \tau] := \{M : \sigma \rightarrow \tau \mid M \text{ closed}\}$$

be the set of closed terms of type $\sigma \rightarrow \tau$. Suppose that for each type symbol $\rho$, there is a function,

$$f_{\rho} : [\rho \rightarrow \sigma] \rightarrow [\rho \rightarrow \tau]$$

with the following properties:

- for any closed terms $M, N : \rho \rightarrow \sigma$, if $\vdash_{\mathcal{L}} M = N$ ($\mathcal{L}$-provably equivalent), then $f_{\rho} M = f_{\rho} N$,
- for any closed terms $M : \mu \rightarrow \nu$ and $N : \nu \rightarrow \sigma$,

$$\vdash_{\mathcal{L}} f_{\mu}(\lambda x : \mu, N (M x)) = \lambda x : \mu, (f_{\nu}(N))(M x).$$

Use the Yoneda embedding of the cartesian closed category of types $\mathbf{C(\mathcal{L})}$ of $\mathcal{L}$ to show that there is a term $F : \sigma \rightarrow \tau$ such that $f_{\rho}$ is induced by composition with $F$, in the sense that, for every closed term $R : \rho \rightarrow \sigma$,

$$\vdash_{\mathcal{L}} f_{\rho}(R) = \lambda x : \rho, F(R x).$$

Show that, moreover, $F$ is unique up to $\mathcal{L}$-provable equivalence.
3. Show that every slice category \textbf{Sets}/X is cartesian closed, by proving it equivalent to the presheaf topos \textbf{Sets}^X. Calculate the exponential of two objects \( A \rightarrow X \) and \( B \rightarrow X \) by transferring across the equivalence.

4. (*) Explicitly determine the graph that is the subobject classifier \( \Omega \) in the topos of graphs (i.e., what are its edges and vertices). How many points \( 1 \rightarrow \Omega \) does it have?