In this homework set you are going to prove the Kripke completeness theorem for the positive fragment of IPC, IPC+. Recall that the propositions are formed from atomic propositions using just the connectives $\top$, $\land$, and $\to$. (NB: the notations and definitions here differ slightly from the book in the hopes of making things clearer.)

Fix a theory $\mathcal{L}$ over IPC+, given by a set of atomic propositions $\Sigma$ and a set of axioms in $\text{Prop}^+(\Sigma)$ (the positive propositions generated by $\Sigma$). That is, $\text{Prop}^+(\Sigma)$ is inductively generated by the following rules:

\[
\begin{align*}
p &\in \Sigma \\
p &\in \text{Prop}^+(\Sigma) \\
\top &\in \text{Prop}^+(\Sigma) \\
p, q &\in \text{Prop}^+(\Sigma) \\
p \land q &\in \text{Prop}^+(\Sigma) \\
p \to q &\in \text{Prop}^+(\Sigma)
\end{align*}
\]

Since $\mathcal{L}$ is fixed we write simply $\vdash p$ if the proposition $p \in \text{Prop}^+(\Sigma)$ is provable from the rules of IPC+ from the axioms of $\mathcal{L}$.

**Definition 1.** A Kripke frame for $\Sigma$ consists of a poset $I$ of “possible worlds,” together with a relation $\models \subseteq I \times \Sigma$ satisfying the following monotonicity condition:

\[
i \leq j \text{ and } i \models p \text{ implies } j \models p. \quad (\star)
\]

Given a Kripke frame for $\Sigma$, we can extend $\models$ to a relation $\models \subseteq I \times \text{Prop}^+(\Sigma)$ as follows (by recursion on propositions in $\text{Prop}^+(\Sigma)$):

1. $i \models \top$ always,
2. $i \models p \land q$ if and only if $i \models p$ and $i \models q$,
3. $i \models p \to q$ if and only if for all $j \geq i$, if $j \models p$, then $j \models q$.

**Question 1.** Prove that the extended relation satisfies the monotonicity condition $(\star)$ for all $p \in \text{Prop}^+(\Sigma)$.

**Definition 2.** We write $I \models p$ if and only if $i \models p$ for all $i \in I$. The Kripke frame $(I, \models)$ is a Kripke model of $\mathcal{L}$ if and only if $I \models p$ for all axioms $p$ of $\mathcal{L}$.

**Theorem 1** (to be proved). For $p \in \text{Prop}^+(\Sigma)$ we have $\vdash p$ in and only if $I \models p$ for all Kripke models of $\mathcal{L}$.
We shall prove this by reducing it to our completeness proof in CCC posets (aka HA\(^-\)s). To this end, note that a Kripke frame on \(I\) can be construed as function

\[
[-] : \Sigma \rightarrow 2^I
\]

from the set of atomic propositions into (the underlying set of) the exponential poset \(2^I\) of monotone maps from \(I\) into the poset \(2 = \{\bot \leq \top\}\). The mapping is determined by the condition

\[
i \models p \text{ if and only if } \llbracket p \rrbracket(i) = \top.
\]

**Question 2.** Prove that for any poset \(I\), the exponential poset \(2^I\) is a CCC poset, and in fact a Heyting algebra, with limits and colimits computed pointwise and implication defined by \((p \Rightarrow q)(i) = \top \in 2\) if and only if for all \(j \geq i\), \(p(j) \leq q(j)\).

The extension of \(\models\) to all of \(\text{Prop}^+(\Sigma)\) becomes, by Question 1, a function

\[
[-] : \text{Prop}^+(\Sigma) \rightarrow 2^I.
\]

So by the UMP for the Lindenbaum-Tarski algebra \(HA^- (\mathcal{L})\), this descends for every Kripke model of \(\mathcal{L}\) to an HA\(^-\) homomorphism

\[
[-] : HA^- (\mathcal{L}) \rightarrow 2^I.
\]

Thus, a Kripke model is just an HA\(^-\) model in a HA\(^-\) of the special form \(2^I\).

For any poset \(A\) we can consider the function \(y : A \rightarrow 2^{A^\text{op}}\) defined by

\[
y(a)(x) = \top \text{ if and only if } x \leq a.
\]

**Question 3.** Show that we can identify elements of \(2^{A^\text{op}}\) with lower sets \(S \subseteq A\), that is, subsets \(S \subseteq A\) such that \(x \leq y \& y \in S\) implies \(x \in S\).

Under this identification, \(y\) maps \(a\) to \(\downarrow(a) := \{x \in A \mid x \leq a\}\).

**Question 4.** Show that for any HA\(^-\), \(A\), the map \(y : A \rightarrow 2^{A^\text{op}}\) is an injective homomorphism of HA\(^-\)s (that is, a monotone map that preserves \(\top\), \(\wedge\) and \(\Rightarrow\)). Now conclude Theorem 1 from completeness in HA\(^-\)s.

**Question 5 (\(*\)).** Show that the map \(y\) doesn’t in general preserve \(\bot\) and \(\lor\) (thus we can’t use this argument to infer completeness for the full IPC).