1. Prove that \textbf{Sets} has all coequalizers by constructing the coequalizer of a parallel pair of functions.

2. Consider the following commutative diagram of categories and functors,

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B & \xrightarrow{G} & C \\
\downarrow{H} & & \uparrow{K} & & \\
D & & & & \\
\end{array}
\]

where, describing the categories as free or finitely presented categories,

- \(A = 2\) = (the arrow) = ( \(\bullet \xrightarrow{f} \bullet\) ),
- \(B = (\mathbb{N} \text{ as monoid category}) = ( \bullet \varnothing e \bullet\) ,
- \(C = \text{(the idempotent)} = ( \bullet \varnothing e \text{ where } e^2 = e\) ,
- \(D = \text{(the parallel pair)} = ( \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet\) ),

and \(F(f) = e, G(e) = e, H(f) = f, K(f) = e, K(g) = e^2\).

Show that \(F\) and \(G\) are regular epimorphisms in \textbf{Cat}, but \(G \circ F\) and \(K\) are not.

3. Given a homomorphism of abelian groups \(f : A \rightarrow B\), define the cokernel \(c : B \rightarrow C\) to be the quotient of \(B\) by the subgroup \(\text{im}(f) \subseteq B\).

(a) Show that the cokernel has the following UMP: \(c \circ f = 0\), and if \(g : B \rightarrow G\) is any homomorphism with \(g \circ f = 0\), then \(g\) factors uniquely through \(c\).

(b) Show that the cokernel is a particular kind of coequalizer, and use cokernels to construct arbitrary coequalizers.

(c) Take the kernel of the cokernel, and show that \(f : A \rightarrow B\) factors through it.
4. (*) With \( f \) and \( c \) as in the previous exercise, prove that the two factorizations of \( f \),

\[
\begin{array}{c}
A \\
\downarrow \text{ker}(c) \\
\downarrow \text{im}(f)
\end{array}
\xrightarrow{f} B 
\xrightarrow{c} C
\]

are isomorphic in the category of factorizations of \( f \) (define this category first!).