1. (Kleisli category) Given a monad \((T, \eta, \mu)\) on a category \(C\), in addition to the Eilenberg-Moore category, we can construct another category \(C_T\) and an adjunction \(F \dashv U\) with \(U : C_T \to C\) such that \(T = UF\), the unit of the adjunction equals the unit \(\eta\) of the monad, and \(\mu = U(\varepsilon_F)\). This category \(C_T\) is called the Kleisli category of the monad, and is defined as follows:

- the objects are the same as those of \(C\),
- an arrow \(f : A \to_T B\) in \(C_T\) is an arrow \(f : A \to TB\) in \(C\),
- the identity arrow \(1^T_A : A \to_T A\) in \(C_T\) is the unit arrow \(\eta_A : A \to TA\) in \(C\),
- for composition, given \(f : A \to_T B\) and \(g : B \to_T C\), the composite \(g \circ_T f : A \to_T C\) is defined to be the composite

\[
\mu_C \circ T(g) \circ f
\]

in \(C\) as indicated in the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{g \circ_T f} & TC \\
\downarrow{f} & & \uparrow{\mu_C} \\
TB & \xrightarrow{T(g)} & T^2 C
\end{array}
\]

Verify that this indeed defines a category, and that there are adjoint functors \(F : C \to C_T\) and \(U : C_T \to C\) giving rise to the monad \(T = UF\), as claimed.

2. Let \(P : \text{Sets} \to \text{Sets}\) be a finitary polynomial functor,

\[
P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n
\]

(thus the coefficient sets \(C_i\), \(i = 0, \ldots, n\), are finite). Show that \(P\) preserves \(\omega\)-colimits.
3. The notion of a coalgebra for an endofunctor $P : S \to S$ on an arbitrary category $S$ is exactly dual to that of a $P$-algebra. Determine the final coalgebra for the endofunctor

$$P(X) = 1 + A \times X$$
on $\text{Sets}$, parametrized by a set $A$. (Hint: recall that the initial algebra consists of finite lists over $A$.)

4. (⋆) Prove that a (strict) monadic functor $U : D \to C$ creates coequalizers of $U$-split pairs, where a parallel pair $f, g : D \rightrightarrows D'$ in $D$ is called $U$-split if

$$U(D) \xrightarrow{U(f)} U(D') \xrightarrow{U(g)}$$

extends to a split coequalizer diagram in $C$ (cf. problem 2 on the midterm, the results of which you may use for this problem).

(To prove this, it suffices to look at $U : C^T \to C$ the forgetful functor from the Eilenberg-Moore category of a monad $T$ on $C$. This accounts for the “strict” qualification to match our strong notion of creation of limits. Other sources, such as the $nLab$, defines creation of limits in a slightly weaker way that is invariant under equivalence of categories, so that the statement holds for any monadic functor with the same proof.)