1. Recall the definition of the **comma category** \((F \downarrow G)\) given functors \(F : C \to E\) and \(G : D \to E\):

   - The objects are triples \((C, D, h)\) where \(C \in C\), \(D \in D\), and \(h : FC \to GD\) in \(E\).
   - The arrows from \((C, D, h)\) to \((C', D', h')\) are pairs of arrows \((f, g)\) where \(f : C \to C'\) in \(C\), \(g : D \to D'\) in \(D\), such that \(h' \circ Ff = Gg \circ h\).

Consider a functor \(U : D \to X\) (between small categories) and an object \(X \in X\). Prove that the following diagram is a pullback square in \(\text{Cat}\):

\[
\begin{array}{ccc}
(X \downarrow U) & \longrightarrow & D \\
\downarrow & & \downarrow U \\
(X \downarrow 1_X) & \longrightarrow & X
\end{array}
\]

Here, \(X\) in the comma categories refers to the functor \(1 \to X\) picking out the object \(X\), the horizontal functors are projections, and \((X \downarrow 1_X)\) is isomorphic to the coslice category \(X/X\).

2. Consider the inclusion functor \(i : P(I) \to \text{Sets}/I\) that takes a subset \(U \subseteq I\) to its inclusion function \(i(U) : U \to I\). Show that this is a functor and that it has a left adjoint \(\sigma : \text{Sets}/I \to P(I)\).

3. In \(\text{Sets}\), given any function \(f : I \to J\), consider the following diagram of functors (Lawvere’s Hyperdoctrine Diagram):

\[
\begin{array}{ccc}
\text{Sets}/I & \xleftarrow{\Pi_f} & \text{Sets}/J \\
\sigma_I & \downarrow i_I & \sigma_J \\
\text{P}(I) & \xleftarrow{f^{-1}} & \text{P}(J)
\end{array}
\]

\[
\begin{array}{cccc}
\Sigma_f & \xrightarrow{j_f} & \forall_f \\
\sigma_J & \downarrow i_J & \sigma_I \\
\text{P}(I) & \xleftarrow{f^{-1}} & \text{P}(J)
\end{array}
\]
Recall that there are adjunctions \( \sigma \dashv i \) (for both \( I \) and \( J \), from problem 2), as well as
\[
\Sigma_f \dashv f^* \dashv \Pi_f
\]
and
\[
\exists_f \dashv f^{-1} \dashv \forall_f
\]
where \( f^* \) is pullback and \( f^{-1} \) is inverse image.

Which of the six possible squares (matching vertical and horizontal arrows in the same position) commute (up to natural isomorphism if necessary), and why?

4. (\(*\)) Use the Adjoint Functor Theorem (as proved in class, and in Mac Lane’s *Categories for the Working Mathematician*, or on the *nLab*) to prove the existence of the free category functor \( F : \textbf{Graphs} \to \textbf{Cat} \), as the left adjoint to the forgetful functor.

Think about how this compares with the explicit construction. (You don’t have to submit your thoughts for grading!)