A curious gap in one-dimensional geometric random graphs between connectivity and the absence of isolated node

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Abstract—One-dimensional geometric random graphs are constructed by distributing \( n \) nodes uniformly and independently on a unit interval and then assigning an undirected edge between any two nodes that have a distance at most \( r_n \). These graphs have received much interest and been used in various applications including wireless networks. A threshold of \( r_n \) for connectivity is known as \( r_n^* = \frac{\ln n}{n} \) in the literature. In this paper, we prove that a threshold of \( r_n \) for the absence of isolated node is \( \frac{\ln n}{2n} \) (i.e., a half of the threshold \( r_n^* \)). Our result shows there is a curious gap between thresholds of connectivity and the absence of isolated node in one-dimensional geometric random graphs; in particular, when \( r_n \) equals \( \frac{c\ln n}{n} \) for a constant \( c \in \left(\frac{1}{2}, 1\right) \), a one-dimensional geometric random graph has no isolated node but is not connected. This curious gap in one-dimensional geometric random graphs is in sharp contrast to the prevalent phenomenon in many other random graphs such as two-dimensional geometric random graphs, Erdős–Rényi graphs, and random intersection graphs, which all of which in the asymptotic sense become connected as soon as there is no isolated node.

Index Terms—Connectivity, geometric random graphs, isolated node, wireless networks, zero-one laws.

I. INTRODUCTION

One-dimensional geometric random graphs have received much attention in the literature \([1,4,6,7,13]\). A one-dimensional geometric random graph is constructed as follows: first, \( n \) nodes are uniformly and independently distributed on a unit interval; and second, an undirected edge is put between any two nodes that have a distance no greater than \( r_n \). These graphs represent the topologies of wireless networks where each wireless node has a transmission range of \( r_n \) and two nodes have to be within the transmission range \( r_n \) for communication (i.e., the well-known disk model \([4]\)). In addition to wireless networks, these graphs have also been used for percolation analysis, cluster analysis, hypothesis testing, and statistical physics \([9]\).

Connectivity of one-dimensional geometric random graphs has been investigated in several work \([1,4,6]\), where a graph is connected if any pair of nodes can find at least one path in between. Our paper studies another graph property highly related to connectivity – the absence of isolated node (i.e., each node has at least one neighboring node). Clearly, any connected graph has no isolated node. The properties of connectivity and the absence of isolated node are particularly important in the application of one-dimensional geometric random graphs to wireless networks as communications between nodes are needed for many tasks.

In one-dimensional geometric random graphs, a threshold of \( r_n \) for connectivity is known as \( r_n^* = \frac{\ln n}{n} \) in the literature \([1,4,6]\). Since in many other random graphs such as two-dimensional geometric random graphs, Erdős–Rényi graphs, and random intersection graphs \([1]\), the respective thresholds for connectivity and for the absence of isolated node are always equivalent \([2,8]\), \([14]\) (all these graphs in the asymptotic sense become connected as soon as there is no isolated node), it is natural to assume that in one-dimensional geometric random graphs, the threshold \( r_n^* = \frac{\ln n}{n} \) of \( r_n \) for connectivity is also a threshold of \( r_n \), for the absence of isolated node. However, this natural assumption is actually incorrect as we prove that \( \frac{1}{2} r_n^* \) (i.e., \( \frac{\ln n}{2n} \)) rather than \( r_n^* \) is a threshold of \( r_n \) for the absence of isolated node in one-dimensional geometric random graphs. In other words, in sharp contrast to many other random graphs mentioned above, one-dimensional geometric random graphs exhibit a curious gap between thresholds of connectivity and the absence of isolated node; in particular, when \( r_n \) equals \( \frac{c\ln n}{n} \) for a constant \( c \in \left(\frac{1}{2}, 1\right) \), a one-dimensional geometric random graph has no isolated node but is not connected.

We use the following definition of graph threshold.

Definition 1. Consider a random graph \( G_n \), where \( n = 1, 2, \ldots \). Let \( s_n \) be a sequence related to \( G_n \). The term \( s_n^* \) is said to be a threshold of \( s_n \) for a monotone property \( \mathcal{P} \) if it holds under the assumptions that

\[
\lim_{n \to \infty} \mathbb{P}[G_n \text{ has } \mathcal{P}] = \begin{cases} 0, & \text{if } c < 1, \\ 1, & \text{if } c > 1. \end{cases}
\]

The rest of the paper is organized as follows. We explain the models and notation in Section II. Section III presents the main results as Theorem 1 and also provides some discussions. Then, we introduce several auxiliary lemmas in Section IV before establishing Theorem 1 in Section V. Section VI reviews related work, and Section VII concludes the paper. The Appendix details the proofs of a few lemmas.

1An Erdős–Rényi graph \([2]\) is constructed by assigning an edge between any pair of nodes independently with the same probability. A two-dimensional geometric random graph \([8]\) is constructed by distributing \( n \) nodes uniformly and independently on a Euclidean plane and then assigning an undirected edge between any two nodes that are within a certain distance. A random intersection graph \([12]\) is constructed by assigning a set of items to each node following the same probabilistic distribution and then putting an undirected edge between any two nodes that share at least a certain number of items.

2For two positive sequences \( x_n \) and \( y_n \), the relation \( x_n \sim y_n \) means \( \lim_{n \to \infty} (x_n/y_n) = 1 \).
II. MODELS AND NOTATION

A. One-dimensional geometric random graphs

We discuss two related models of one-dimensional geometric random graphs below.

First, we consider $n$ nodes uniformly and independently distributed on a unit interval $L = [0,1]$, i.e., the locations $X_1, X_2, \ldots, X_n$ follow independent uniform distributions in $[0,1]$. The distance of two nodes at locations $X_i$ and $X_j$ is given by

$$d(L)(X_i, X_j) = |X_i - X_j|.$$  

An undirected edge is constructed between any two nodes that have a distance at most $r_n$ so that two nodes at locations $X_i$ and $X_j$ have an edge in between if and only if $d(L)(X_i, X_j) \leq r_n$ (i.e., $|X_i - X_j| \leq r_n$). The resulting random graph on the unit interval $L$ is denoted by $G^{(L)}(n, r_n)$.

The unit interval has boundary effect in that two nodes close to the two endpoints of $L$ often do not have an edge in between. As in many analyses of geometric random graphs, we also investigate the case where the boundary effect is not considered. Specifically, we consider a circle $C = [0,1]$ of unit circumference as follows. $n$ nodes are uniformly and independently distributed on the unit circle $C$. The nodes have locations determined by the lengths $X_1, X_2, \ldots, X_n$ of the clockwise arcs taken with respect to some reference point; again, the locations $X_1, X_2, \ldots, X_n$ follow independent uniform distributions in $[0,1]$. We measure the distance between any two nodes by the length of the smallest arc between the two nodes, i.e., the distance between two nodes at locations $X_i$ and $X_j$ is given by

$$d(C)(X_i, X_j) = \min(|X_i - X_j|, 1 - |X_i - X_j|).$$

An undirected edge exists between any two nodes that have a distance at most $r_n$ so that two nodes at locations $X_i$ and $X_j$ have an edge in between if and only if $d(C)(X_i, X_j) \leq r_n$ (i.e., $|X_i - X_j| \leq r_n$ or $|X_i - X_j| \geq 1 - r_n$). The resulting random graph on the unit circle $C$ is denoted by $G^{(C)}(n, r_n)$.

B. Notation

Throughout the paper, asymptotic statements are understood with $\rightarrow \infty$. We use the standard asymptotic notation $O(\cdot), o(\cdot), \Omega(\cdot), \omega(\cdot), \Theta(\cdot), \sim$. Also, $\mathbb{P}[\cdot]$ denotes the probability that event $\mathcal{E}$ occurs. An event happens asymptotically almost surely if its probability converges to 1 as $n \rightarrow \infty$. For events or variables, we use the superscripts $(C)$ and $(L)$ to denote the cases on the unit circle $C$ and the unit interval $L$, respectively. We omit the superscripts $(C)$ and $(L)$ when the notation applies to both cases of $C$ and $L$.

III. RESULTS AND DISCUSSIONS

We present the main results and discuss them along with related results in prior work.

A. The absence of isolated node in one-dimensional geometric random graphs

Theorem 1 below establishes results on the absence of isolated node in one-dimensional geometric random graphs.

**Theorem 1.** In a graph $G^{(C)}(n, r_n)$ or a graph $G^{(L)}(n, r_n)$, with a sequence $\alpha_n$ defined by

$$r_n = \frac{\ln n + \alpha_n}{2n},$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}[G^{(C)}(n, r_n) \text{ has no isolated node.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty, \end{cases}$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}[G^{(L)}(n, r_n) \text{ has no isolated node.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \end{cases}$$

The results (2a) and (2b) (resp., (3a) and (3b) in Theorem 1) present a zero–one law in a graph $G^{(C)}(n, r_n)$ (resp. $G^{(C)}(n, r_n)$) for the absence of isolated node. We present below Corollary 1 as a simple corollary of Theorem 1 to determine a threshold of $r_n$ for the absence of isolated node.

**Corollary 1** (A corollary of Theorem 1). In a graph $G^{(C)}(n, r_n)$ or a graph $G^{(L)}(n, r_n)$, if there is a positive constant $c$ such that

$$r_n \sim \frac{c \ln n}{n},$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}[G^{(C)}(n, r_n) \text{ has no isolated node.}] = \begin{cases} 0, & \text{if } c < \frac{1}{2}, \\ 1, & \text{if } c > \frac{1}{2}, \end{cases}$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}[G^{(L)}(n, r_n) \text{ has no isolated node.}] = \begin{cases} 0, & \text{if } c < \frac{1}{2}, \\ 1, & \text{if } c > \frac{1}{2}. \end{cases}$$

From Definition 1 and Corollary 1 in both $G^{(C)}(n, r_n)$ and $G^{(L)}(n, r_n)$, a common threshold of $r_n$ for the absence of isolated node is $\frac{\ln n}{2n}$. Corollary 1 clearly follows from Theorem 1 since we obtain from (1) and (2) that

$$\alpha_n = 2nr_n - \ln n = 2c \ln n \cdot [1 + o(1)] - \ln n = (2c - 1) \ln n + o(\ln n) \rightarrow \begin{cases} -\infty \text{ as } n \rightarrow \infty, & \text{if } c < \frac{1}{2}, \\ \infty \text{ as } n \rightarrow \infty, & \text{if } c > \frac{1}{2}. \end{cases}$$

B. A gap between connectivity and the absence of isolated node in one-dimensional geometric random graphs

We now explain a curious gap between connectivity and the absence of isolated node in both $G^{(C)}(n, r_n)$ and $G^{(L)}(n, r_n)$. Theorem 1 and Corollary 1 above provide zero–one laws for the absence of isolated node, while analogous zero–one laws for connectivity are given by Lemmas 1 and 2 below, respectively. Lemma 1 is reproduced from [4, Theorem 2.1]. Lemma 2 is from [1, Theorem 1] or [9, Theorem 2.2], or can be seen as a corollary of Lemma 1 in a way similar to Corollary 1 as a corollary of Theorem 1.

**Lemma 1** ([4, Theorem 2.1]). In a graph $G^{(C)}(n, r_n)$ or a graph $G^{(L)}(n, r_n)$, with a sequence $\beta_n$ defined by

$$r_n = \frac{\ln n + \beta_n}{n},$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}[G^{(C)}(n, r_n) \text{ is connected.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \beta_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \beta_n = \infty, \end{cases}$$
and
\[ \lim_{n \to \infty} \mathbb{P} \left[ G^L(n, r_n) \text{ is connected.} \right] = \begin{cases} 0, & \text{if } \lim_{n \to \infty} \beta_n = -\infty, \\ 1, & \text{if } \lim_{n \to \infty} \beta_n = \infty. \end{cases} \]

Lemma 2 (Theorem 1, 6 Theorem 2.2) or a corollary of Lemma 1. In a graph \( G^C(n, r_n) \) or a graph \( G^L(n, r_n) \), if there is a positive constant \( c \) such that
\[ r_n \sim \frac{c \ln n}{n}, \]
then
\[ \lim_{n \to \infty} \mathbb{P} \left[ G^C(n, r_n) \text{ is connected.} \right] = \begin{cases} 0, & \text{if } c < 1, \\ 1, & \text{if } c > 1. \end{cases} \]

Both Lemmas 1 and 2 present zero-one laws in graphs \( G^C(n, r_n) \) and \( G^L(n, r_n) \) for connectivity. Furthermore, from Definition 1 and Lemma 2 in both \( G^C(n, r_n) \) and \( G^L(n, r_n) \), a common threshold of \( r_n \) for connectivity is \( \frac{\ln n}{n} \), which we denote by \( r^* \). As mentioned above, in both \( G^C(n, r_n) \) and \( G^L(n, r_n) \), there is a curious gap between connectivity and the absence of isolated node; specifically, as formally given in Corollary 1 below which summarizes Corollary 1 and Lemma 2, for \( r_n \sim \frac{c \ln n}{n} \) with a constant \( c \in \left( \frac{1}{2}, 1 \right) \), asymptotically almost surely both \( G^C(n, r_n) \) and \( G^L(n, r_n) \) have no isolated node, but are not connected. This gap in one-dimensional geometric random graphs is not observed in many other random graphs such as two-dimensional geometric random graphs, Erdős–Rényi graphs, and random intersection graphs, all of which become connected as soon as there is no isolated node.

Corollary 2 (A summary of Corollary 1 and Lemma 2). In a graph \( G^C(n, r_n) \) or a graph \( G^L(n, r_n) \), if there is a positive constant \( c \) such that
\[ r_n \sim \frac{c \ln n}{n}, \]
then
(i) If \( c < \frac{1}{2} \), then asymptotically almost surely both \( G^C(n, r_n) \) and \( G^L(n, r_n) \) have at least one isolated node.
(ii) If \( \frac{1}{2} < c < 1 \), then asymptotically almost surely both \( G^C(n, r_n) \) and \( G^L(n, r_n) \) have no isolated node, but are not connected.
(iii) If \( c > 1 \), then asymptotically almost surely both \( G^C(n, r_n) \) and \( G^L(n, r_n) \) are connected.

Corollary 2 clearly follows from Corollary 1 and Lemma 2 so its proof is omitted.

IV. AUXILIARY LEMMAs

We provide a few lemmas which are used to establish Theorem 1. The proofs of the lemmas are straightforward and thus omitted here (the proofs can be found in Section VI of the full version 15).

Lemma 3. The following results hold.
(i) For two graphs \( G^C(n, r_n^{\text{a}}) \) and \( G^C(n, r_n^{\text{b}}) \), it holds for each \( n \) that if \( r_n^{\text{a}} \leq r_n^{\text{b}} \), then there exists a graph coupling \( \hat{G} \) under which \( G^C(n, r_n^{\text{a}}) \) is a spanning subgraph of \( G^C(n, r_n^{\text{b}}) \).
(ii) For two graphs \( G^C(n, r_n^{\text{a}}) \) and \( G^C(n, r_n^{\text{b}}) \), it holds for each \( n \) that if \( r_n^{\text{a}} \leq r_n^{\text{b}} \), then there exists a graph coupling under which \( G^C(n, r_n^{\text{a}}) \) is a spanning subgraph of \( G^C(n, r_n^{\text{b}}) \).

Lemma 4. The following results hold under \( r_n = \frac{\ln n + \alpha_n}{2n} \).
(i) If \( \lim_{n \to \infty} \alpha_n = -\infty \), there exists \( r_n = \frac{\ln n + \alpha_n}{2n} \) with \( \lim_{n \to \infty} \alpha_n = -\infty \) and \( \alpha_n = -o(\ln n) \) such that there exists a graph coupling under which \( G^C(n, r_n) \) is a spanning subgraph of \( G^C(n, \tilde{r}_n) \) and \( G^L(n, r_n) \) is a spanning subgraph of \( G^L(n, \tilde{r}_n) \).
(ii) If \( \lim_{n \to \infty} \alpha_n = \infty \), there exists \( \tilde{r}_n = \frac{\ln n + \alpha_n}{2n} \) with \( \lim_{n \to \infty} \alpha_n = \infty \) and \( \alpha_n = o(\ln n) \) such that there exists a graph coupling under which \( G^C(n, r_n) \) is a spanning subgraph of \( G^C(n, \tilde{r}_n) \) and \( G^L(n, r_n) \) is a spanning subgraph of \( G^L(n, \tilde{r}_n) \).

Lemma 5. For each \( n \), there exists a graph coupling under which \( G^C(n, r_n) \) is a spanning subgraph of \( G^C(n, r_n) \).

Lemma 6 (Straightforward from 16 Lemma 3-(b), page 21). If \( r_n = \frac{\ln n + \alpha_n}{2n} \) and \( |\alpha_n| = o(\ln n) \), then it holds for any constant \( c \) that
\[ (1 - cr_n)^n \sim (1 - cr_n)^{-n} \sim n^{-c/2}e^{-cn/2} = o(1). \]

V. ESTABLISHING THEOREM 1

From Lemma 5, 2a implies 2a, and 3b implies 2b. Hence, we only need to prove 2a and 3b to establish Theorem 1.

A. Constraining \( |\alpha_n| \) as \( o(\ln n) \)

From Lemma 4 and the fact that the absence of isolated node is a monotone increasing graph property, we can introduce an auxiliary condition \( \alpha_n = o(\ln n) \) in proving Theorem 1. From (1) and \( |\alpha_n| = o(\ln n) \), it follows that if \( r_n = \frac{\ln n + \alpha_n}{2n} \) and \( |\alpha_n| = o(\ln n) \), then it holds for any constant \( c \) that
\[ r_n \sim \frac{\ln n}{2n}, \]
which further implies
\[ r_n < \frac{1}{4} \text{ for all } n \text{ sufficiently large}. \]

B. Method of moments

Let \( v_x \) and \( v_y \) be two typical and different nodes in a graph \( G^C(n, r_n) \) or a graph \( G^L(n, r_n) \). We use \( x \) (resp., \( y \)) to denote the position of node \( v_x \) (resp., \( v_y \)). Let \( I_x \) (resp., \( I_y \)) be the event that node \( v_x \) (resp., \( v_y \)) is isolated.

As used by Rybarczyk 10, 11, a coupling of two random graphs \( G_1 \) and \( G_2 \) means a probability space on which random graphs \( G_1 \) and \( G_2 \) are defined such that \( G_1 \) and \( G_2 \) have the same distributions as \( G_1 \) and \( G_2 \), respectively. If \( G_1 \) is a spanning subgraph (resp., spanning supergraph) of \( G_2 \), we say that under the coupling, \( G_1 \) is a spanning subgraph (resp., spanning supergraph) of \( G_2 \), which yields that for any monotone increasing property \( I \), the probability of \( G_1 \) having \( I \) is at most (resp., at least) the probability of \( G_2 \) having \( I \).
As mentioned above, to show Theorem 1 it suffices to prove only (2a) and (3b). From the method of of moments (9), (16), (2a) follows once we show both
\[
\lim_{n \to \infty} \left( n \mathbb{P}[I_x^n(\mathcal{C})] \right) = \infty \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty. \quad (7)
\]
and
\[
\lim_{n \to \infty} \frac{n \mathbb{P}[I_x^n(\mathcal{C})] \cap I_y^n(\mathcal{C})]}{\mathbb{P}[I_x^n(\mathcal{C})]} = 1 \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty. \quad (8)
\]
Moreover, (3b) follows once we prove
\[
\lim_{n \to \infty} \left( n \mathbb{P}[I_x^n(\mathcal{C})] \right) = 0 \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = \infty. \quad (9)
\]
Below we will prove (7) and (9). As explained above, (7) establishes (3a) and thus (3a), and (9) establishes (3b) and hence (2b). We suppress the subscript \(n\) of \(r_n\) and use \(r\) for simplicity.

C. Proving (2)

With \(Y\) being a random variable that is uniformly distributed in \([0, 1]\), it follows that
\[
\mathbb{P}[I_x^n(\mathcal{C})] = \int_0^1 \left\{ \mathbb{P}[d(x, Y) > r] \right\}^{n-1} dx, \quad (10)
\]
where
\[
\mathbb{P}[d(x, Y) > r] = 1 - 2r, \quad \text{for} \quad x \in [0, 1]. \quad (11)
\]
Using (11) in (10), we obtain
\[
\mathbb{P}[I_x^n(\mathcal{C})] = (1 - 2r)^{n-1}. \quad (12)
\]
From (12) and Lemma 6 it follows that
\[
n \mathbb{P}[I_x^n(\mathcal{C})] \sim e^{-\alpha n} \to \infty \quad \text{as} \quad n \to \infty \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty,
\]
which completes proving (2).

D. Proving (13)

With \(Z\) being a random variable that is uniformly distributed in \([0, 1]\), it follows that
\[
\mathbb{P}[I_x^n(\mathcal{C}) \cap I_y^n(\mathcal{C})] = \int_{0}^{1} \left\{ \mathbb{P}[d(x, Y) > r] \right\} (\mathbb{P}[d(x, Z) > r] \cap (d(x, Y) > r))^n-2 dx, \quad (13)
\]
We use \(a\) to denote \(d(x, y)\), and obtain
\[
\mathbb{P}[d(x, Z) > r] \cap (d(x, Y) > r) = \begin{cases} 1 - 4r, & \text{if} \quad a \in [2r, \frac{1}{2}] \\ 1 - a - 2r, & \text{if} \quad a \in [0, 2r). \end{cases} \quad (14)
\]
For each fixed \(x\), we have the following observations: on the one hand, the range of \(y \in [0, 1]\) satisfying \(d(x, y) > r\) yields \(a \in \left[ r, \frac{1}{2} \right] \); on the other hand, each \(a \in \left[ r, \frac{1}{2} \right] \) corresponds to different \(y\): \(x - a + 1 \leq x < 0\) and \(x + a - 1 \leq x + a \leq 1\), where \(1[\mathcal{E}]\) with an event \(\mathcal{E}\) denotes an indicator variable: \(1[\mathcal{E}] = 1\) (resp., 0) if \(\mathcal{E}\) occurs (resp., does not occur). Therefore, with \(f(a)\) denoting the right hand side of (14), we obtain from (14) that
\[
\mathbb{P}[I_x^n(\mathcal{C}) \cap I_y^n(\mathcal{C})] = \int_{0}^{1} \left\{ dx \cdot 2 \int_{r}^{1/2} f(a) da \right\} = 2 \int_{r}^{1/2} f(a) da,
\]
which with (14) further yields
\[
\mathbb{P}[I_x^n(\mathcal{C}) \cap I_y^n(\mathcal{C})] = 2 \left[ \int_{2r}^{1/2} (1 - 4r)^{n-2} dx + \int_{r}^{1} (1 - a - 2r)^{n-2} dx \right]
= (1 - 4r)^{n-1} + \frac{2}{n-1} \left[ (1 - 3r)^{n-1} - (1 - 4r)^{n-1} \right]. \quad (15)
\]
From (12) and (15), it follows that
\[
\mathbb{P}[I_x^n(\mathcal{C}) \cap I_y^n(\mathcal{C})] = \frac{(1 - 4r)^{n-1}}{(1 - 2r)^{2(n-1)} + \frac{2}{n-1} \left[ (1 - 3r)^{n-1} - (1 - 4r)^{n-1} \right]}. \quad (16)
\]
To evaluate the right hand side of (16), we use Lemma 6 to derive
\[
(1 - 4r)^{n-1} \sim n^{-1/2} e^{-\alpha n}, \quad (17)
(1 - 3r)^{n-1} \sim n^{-3/2} e^{-3\alpha n/2}, \quad (18)
\]
and
\[
(1 - 4r)^{n-1} \sim n^{-2} e^{-2\alpha n}. \quad (19)
\]
From (17) and (19), it is clear that
\[
\frac{2}{n-1} \left[ (1 - 3r)^{n-1} - (1 - 4r)^{n-1} \right] \to 1, \quad \text{as} \quad n \to \infty. \quad (20)
\]
We also obtain from (17) and (18) that
\[
\frac{2}{n-1} \left[ (1 - 3r)^{n-1} - (1 - 4r)^{n-1} \right] \to 2 n^{-1/2} e^{3\alpha n/2}. \quad (21)
\]
Substituting (20) and (21) into (16), we further have
\[
\mathbb{P}[I_x^n(\mathcal{C}) \cap I_y^n(\mathcal{C})] = 1 + o(1) + 2 n^{-1/2} e^{3\alpha n/2} \cdot [1 \pm o(1)] \to 1 \quad \text{as} \quad n \to \infty \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty,
\]
which completes proving (13).

E. Proving (9)

With \(Y\) being a random variable that is uniformly distributed in \([0, 1]\), it follows that
\[
\mathbb{P}[I_y^n(\mathcal{C})] = \int_{0}^{1} \left\{ \mathbb{P}[d(x, Y) > r] \right\}^{n-1} dx, \quad (22)
\]
where
\[
\mathbb{P}[d(x, Y) > r] = \begin{cases} 1 - 2r, & \text{if} \quad x \in (r, 1 - r) \\ 1 - x - r, & \text{if} \quad x \in [0, r], \\ x - r, & \text{if} \quad x \in [1 - r, 1). \end{cases} \quad (23)
\]
Applying (22) to (23), we obtain
\[
\mathbb{P}[I_y^n(\mathcal{C})] = \int_{r}^{1-r} (1 - 2r)^{n-1} dx + \int_{0}^{r} (1 - x - r)^{n-1} dx
\]
\[
+ \int_{1-r}^{1} (x - r)^{n-1} dx
= (1 - 2r)^{n-1} + 2 \cdot n^{-1} [(1 - r)^{n-1} - (1 - 2r)^{n-1}]. \quad (24)
\]
From (24) and Lemma 6 it follows that
\[
n \mathbb{P}[I_x^n(\mathcal{C})] \sim e^{-\alpha n} \cdot [1 \pm o(1)] + o(1) \sim \to \infty \quad \text{as} \quad n \to \infty \quad \text{if} \quad \lim_{n \to \infty} \alpha_n = -\infty,
\]
which completes proving (9).
Recall that an event happens asymptotically almost surely if its probability converges to 1 as \( n \to \infty \). Recall that \( G^{(C)}(n, r_n) \) denotes a one-dimensional geometric random graph on a unit interval \( \mathcal{L} \). Appel and Russo [1], and Muthukrishnan and Pandurangan [6] show a zero-one law as follows (referred to as result (a)): if \( r_n \sim \frac{c \ln n}{n} \) for a positive constant \( c \), then \( G^{(C)}(n, r_n) \) is disconnected asymptotically almost surely if \( c < 1 \), and is connected asymptotically almost surely if \( c > 1 \). Han and Makowski [4] improve the above zero-one law to obtain the following result (referred to as result (b)) which also covers the case of \( c = 1 \): if \( r_n \sim \frac{c \ln n}{n} \) (i.e., \( r_n \sim \frac{\ln n}{n} \)); if \( r_n = \frac{\ln n + \beta_n}{n} \) for a sequence \( \beta_n \), then \( G^{(C)}(n, r_n) \) is disconnected asymptotically almost surely if \( \lim_{n \to \infty} \beta_n = -\infty \), and is connected asymptotically almost surely if \( \lim_{n \to \infty} \beta_n = \infty \). The result (b) above covers \( r_n \sim \frac{c \ln n}{n} \) since \( \beta_n \) can still converge to \(-\infty \) or \( \infty \) when \( \beta_n = \pm (\ln n) \), which with \( r_n = \frac{\ln n + \beta_n}{n} \) yield \( r_n \sim \frac{\ln n}{n} \).\( \beta_n \) can be \( -\Theta(\ln \ln n) - \Theta(\ln \ln n) \), and in other words, the result (b) above by Muthukrishnan and Pandurangan [6] is more fine-grained than the result (a) by Appel and Russo [1], and Muthukrishnan and Pandurangan [6]. In this paper, we establish fine-a grained zero-one law similar to the result (a) for the absence of isolated node: if \( r_n = \frac{\ln n + \alpha_n}{n} \) for a sequence \( \alpha_n \), then \( G^{(C)}(n, r_n) \) has at least one isolated node asymptotically almost surely if \( \lim_{n \to \infty} \alpha_n = -\infty \), and has no isolated node asymptotically almost surely if \( \lim_{n \to \infty} \alpha_n = \infty \). Moreover, we also consider \( G^{(C)}(n, r_n) \) on a unit circle \( \mathcal{C} \) without the boundary effect and show the corresponding result is the same as that of \( G^{(C)}(n, r_n) \).

Two-dimensional geometric random graphs have also been studied in the literature [3, 5, 8, 12]. In a two-dimensional geometric random graph discussed below, \( n \) units are uniformly and independently deployed in a network area \( A \) and there exists an edge between two nodes if and only if their distance is no greater than \( r_n \) (\( r_n \) is the transmission range in wireless network applications). Let \( H^{(A)}(n, r_n) \) denote the induced graph. Gupta and Kumar [3] establish that with \( D \) as a disk of unit area, if \( \pi r_n^2 = \frac{\ln n + \delta_n}{n} \), graph \( H^{(D)}(n, r_n) \) is asymptotically almost surely connected if and only if \( \lim_{n \to \infty} \delta_n = \infty \). Penrose [8] proves that if \( \pi r_n^2 = \frac{\ln n + (k-1) \ln \ln n + \delta_n}{n} \) with \( \lim_{n \to \infty} \delta_n = \delta^* \in [-\infty, \infty] \), then graph \( H^{(T)}(n, r_n) \) on a torus \( T \) of unit area has a probability of \( e^{-\frac{T - \pi r_n^2}{\pi}} \) being \( k \)-connected, leading to the result that such graph is asymptotically almost surely \( k \)-connected (resp., not \( k \)-connected) if \( \lim_{n \to \infty} \delta_n = \infty \) (resp., \( \lim_{n \to \infty} \delta_n = -\infty \)), where the torus \( T \) eliminates the boundary effect (a node on \( T \) "exists" the area from one side appears as reentering from the opposite side). The boundary effect of a square \( S \) of unit area is that a circle with radius \( r_n \) centered a point near the boundary of \( S \) may have a part falling outside of \( S \), so a node close to one side and another node close to the opposite side may not have an edge in between on the square \( S \), but may do on the torus \( T \) because of wrap-around connections on torus topologies. A graph is said to be \( k \)-connected if any two nodes have at least \( k \) internally node-disjoint path(s) in between; and an equivalent definition is that the removal of any \( (k - 1) \) nodes does not disconnect the graph [8]. Penrose [8] also considers \( k \)-connectivity in graph \( H^{(S)}(n, r_n) \) on the square \( S \); and the formula of \( r_n \) is given later by Li et al. [5] as well as by Wan and Yi [12]. Li et al. [5] prove that with \( k \geq 2 \), to have graph \( H^{(S)}(n, r_n) \) asymptotically \( k \)-connected with probability at least \( e^{\frac{T - \pi r_n^2}{\pi}} \) for some \( \delta \), a sufficient condition is that the term \( \pi r_n^2 \) is at least \( \ln n + (2k - 3) \ln \ln n + 2\delta \); and a necessary condition is that \( \pi r_n^2 \) is no less than \( \ln n + (k - 1) \ln \ln n + \delta \). For \( k \geq 2 \), Wan et al. [12] determine the exact formula of \( r_n \) such that graph \( H^{(S)}(n, r_n) \) or \( H^{(D)}(n, r_n) \) is asymptotically \( k \)-connected for \( k \geq 2 \) with a certain probability.

VI. RELATED WORK

Asymptotic almost sure connectivity

In this paper, we establish fine-grained zero-one law for the absence of isolated node, as prior work gives \( r_n = \frac{\ln n}{n} \) as a threshold of \( r_n \) for connectivity in one-dimensional geometric random graphs. When \( r_n \) equals \( \frac{\ln n}{n} \) for a constant \( c \in (\frac{1}{2}, 1) \), a one-dimensional geometric random graph has no isolated node but is not connected. This curious gap is not observed in many other random graphs such as two-dimensional geometric random graphs, Erdős–Rényi graphs, and random intersection graphs, all of which have the same thresholds for connectivity and the absence of isolated node.

REFERENCES