The textbook for this course is *Dynamic Asset Pricing Theory, 3rd ed.* by Darrell Duffie. Dr. Danilova’s office hours are Monday from 15:30 to 16:30 and Wednesday from 17:00 to 18:00 in Doherty Hall 4305.

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Utility

Please note that in this course we will not focus on derivative pricing or on agent utility maximization problems. We will focus on (among other things) the fundamental theorems of asset pricing:

I. A market has no arbitrage if and only if there exists an equivalent martingale measure; and
II. A market is complete if and only if there is at most one such measure.

1 Utility

Utility is a measure of agent satisfaction from consumption. It seems clear that, in the absence of any further information, most people would prefer the consumption vector \( \langle 1, 0.9 \rangle \) to \( \langle 0, 0.1 \rangle \).

1.1 Von Neumann-Morgenstern utility

Fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). \(\Omega\) is to be interpreted as the collection possible outcomes of the world one time-period from today. Let \(X\) be a collection of functions \(X: \Omega \rightarrow A\), where \(A = \{a_1, \ldots, a_n\}\) is a finite set and each \(X\) is to be interpreted as a (one-period) consumption strategy.

Let \(Q\) denote the collection of all probability measures on \(A\). We can and do identify \(Q\) with the set \(\{q \in \mathbb{R}^n_+ | \sum_i q_i = 1\}\), the identification being taking the Radon-Nikodym density with respect to normalized counting measure.

Suppose there is a binary relation \(\succeq\) on \(Q\), to be interpreted as preference. The following axioms will play a role presently.

(A1) \(\succeq\) is a linear order on \(Q\).

(A2) Substitution: \(\forall p, q, r \in Q \forall \lambda \in [0, 1] \quad p \succ q \implies \lambda p + (1 - \lambda) r \succ \lambda q + (1 - \lambda) r\)

(A3) Archimedean: \(\forall p, q, r \in Q \exists \alpha, \beta \in [0, 1] \quad p \succ q \succ r \implies \exists ! \lambda^* \in [0, 1] q \sim \lambda^* p + (1 - \lambda^*) r\)

Warning: Though these conditions are natural mathematically, in the real world preferences may not satisfy these axioms. For example, see http://en.wikipedia.org/wiki/Allais_paradox.

1.1.1 Lemma. If \(\succeq\) satisfies (A1), (A2), and (A3) then

(i) \(p \succ q, 0 < \alpha < \beta < 1 \implies \beta p + (1 - \beta) q \succ \alpha p + (1 - \alpha) q\);
(ii) \(p \succ q \succ r \implies \exists ! \lambda^* \in [0, 1] q \sim \lambda^* p + (1 - \lambda^*) r\);
(iii) \(p \succ q, r \succ s, \lambda \in [0, 1] \implies \lambda p + (1 - \lambda) r \succ \lambda q + (1 - \lambda) s\);
(iv) \(p \sim q, \lambda \in [0, 1] \implies p \sim \lambda p + (1 - \lambda) q\);
(v) \(p \sim q, \lambda \in [0, 1] \implies \forall r \in Q \lambda p + (1 - \lambda) r \sim \lambda q + (1 - \lambda) r\).

A binary relation \(\succeq\) on \(Q\) induces a binary relation on \(\mathcal{X}\), namely \(X \succeq Y\) if \(P_X \succeq P_Y\), where \(P_X\) denotes the law of \(X\), the probability measure on \(A\) defined by \(P_X[ B ] := \mathbb{P}[X \in B]\).
1.1.2 Theorem. There is a function \( u : A \rightarrow \mathbb{R} \) such that
\[
p \succcurlyeq q \iff \mathbb{E}^p[u] \geq \mathbb{E}^q[u]
\]
if and only if the binary relation \( \succcurlyeq \) on \( Q \) satisfies (A1), (A2), and (A3).

**Proof:** Since \( A \) is finite there are \( a, b \in A \) such that \( \delta_a \succcurlyeq \delta_c \succcurlyeq \delta_b \) for all \( c \in A \). But then \( \delta_a \succcurlyeq q \succcurlyeq \delta_b \) for all \( q \in Q \), since elements of \( Q \) are finite convex combinations of Dirac measures. If \( \delta_a \sim \delta_b \) then there is nothing to prove. If \( \delta_a \succcurlyeq \delta_b \) then for each \( q \in Q \) there is a unique \( \lambda^* \in [0, 1] \) such that \( \delta_a \sim \lambda^* \delta_a + (1 - \lambda^*) \delta_b \). Define \( H(q) := \lambda^* \). Then \( H(p) \geq H(q) \) if and only if \( p \succcurlyeq q \), and if \( H \) is convex linear then we are done, taking \( u(c) := H(\delta_c) \) for \( c \in A \). To this end let \( p, q \in Q \) and \( \lambda \in (0, 1) \).

\[
\begin{align*}
H(\lambda p + (1 - \lambda) q) &= H(\lambda (H(p) \delta_a + (1 - H(p)) \delta_b) + (1 - \lambda) (H(q) \delta_a + (1 - H(q)) \delta_b)) \\
&= H((\lambda H(p) + (1 - \lambda))H(q)) \delta_a + (1 - (\lambda H(p) + (1 - \lambda))H(q)) \delta_b \\
&= \lambda H(p) + (1 - \lambda) H(q).
\end{align*}
\]

\( \square \)

1.1.3 Example (Lexicographic order). Take \( \Omega = \{0, 1\} \) and \( X = [0, 1]^2 \), so that \( A = [0, 1] \). Define \( \succcurlyeq \) by
\[
(x_1, y_1) \succcurlyeq (x_2, y_2) \iff x_1 > x_2 \lor (x_1 = x_2 \land y_1 > y_2).
\]
This preferences is not representable by a utility function.

Problems begin to arise when \( A \) is not finite. Even if \( A \) is countable, if the function \( u \) is not bounded then for each \( n \in \mathbb{N} \) there is \( b_n \in A \) such that \( u(b_n) \geq 2^n \). But \( q := \sum_{n \in \mathbb{N}} \frac{1}{n!} b_n \in Q \) and we have \( \mathbb{E}^q[u] = \infty \). The induced order does not satisfy axiom (A3). One possible solution would be to require all measures in \( Q \) have finite first moment and consider only concave functions \( u \).

To this end, let \( A \subseteq \mathbb{R} \) be an arbitrary set of consumption levels and let \( \mathcal{A} \) be \( \sigma \)-algebra on \( A \). We require that our consumption strategies \( X : \Omega \rightarrow A \) be \( \mathcal{F}/\mathcal{A} \)-measurable, and define \( P_X[C] := \mathbb{P}[X \in C] \) for \( C \in \mathcal{A} \). We require of \( A \) the following.

(i) \( \forall c \in A \left\{ c \right\} \in \mathcal{A} \);
(ii) \( \forall c_1, c_2 \in A \left[ c_1, c_2 \right] := \left\{ c \mid \delta_{c_2} \preccurlyeq \delta_c \preccurlyeq \delta_{c_1} \right\} \in \mathcal{A} \).

i.e. \( A \) contains intervals. Let \( Q \) be a collection of probability measures on \( (A, \mathcal{A}) \). We require of \( Q \) the following.

(i) \( \forall c \in A \delta_c \in Q \);
(ii) \( \forall q \in Q \forall C \in \mathcal{A} q(C) > 0 \Rightarrow q_C := q(\cdot \cap C)/q(C) \in Q \);
(iii) \( Q \) is closed under finite convex combinations.

Let \( \succcurlyeq \) be a binary relation on \( Q \). We require, as before, (A1), (A2), and (A3), and additionally,
Risk Aversion

(A4) \( \forall p, q \in Q \forall C \in A \)

\[ q(C) = 1 \implies (\forall c \in C \ p \succeq \delta_c \implies p \succeq q) \land (\forall c \in C \ \delta_c \succeq p \implies q \succeq p) \]

(A5) \( \forall q \in Q \forall c \in A \exists d_u, d_\ell \in A \)

\[ (q \succeq \delta_c \implies q_{(-\infty,d_u]} \succeq \delta_c) \land (\delta_c \succeq q \implies \delta_c \succeq q_{(d_\ell,\infty)}) \]

1.1.4 Theorem. Let \( A \subseteq \mathbb{R} \) be an interval, \( A \) be the Borel \( \sigma \)-algebra, and let \( Q \) be a collection of probability measures on \( A \) satisfying the requirements (i), (ii), and (iii). Let \( \succsim \) be a binary relation on \( Q \). There is a function \( u : A \to \mathbb{R} \) such that \( \forall q \in Q \mathbb{E}^{|u|} < \infty \) and \( p \succsim q \iff \mathbb{E}^p[u] \geq \mathbb{E}^q[u] \) if and only if \( \succsim \) satisfies (A1)–(A5).

Requirement (A5) may be dropped if \( Q \) is closed under countable convex combinations. Notice that the von Neumann-Morgenstern utility function \( u \) depends on \( Q \), whereas in Theorem 1.1.2 the utility function is universal.

Proof: Fix measures \( p \) and \( q \) such that \( p \succ q \) (if there is no such pair then there is nothing to prove). For any \( r \in [q,p] \) there is a unique \( \lambda^* \) such that \( r \sim \lambda^* p + (1 - \lambda^*) q \). Define \( H(r) := \lambda^* \). As in the proof of Theorem 1.1.2, \( H \) is convex linear. If \( [q_1,p_1] \) extends \( [q,p] \) and \( H_1 \) is the associated function defined as above then, for all \( r \in [q,p] \),

\[ H(r) = \frac{H_1(r) - H_1(q)}{H_1(p) - H_1(q)} \]

by uniqueness. For \( r \in Q \) define \( H(r) \) to be the value of the unique extension of \( H \) to \( [r,p] \) or \( [q,r] \), as appropriate. For \( a \in A \) define \( u(a) := H(\delta_a) \). It remains to show for all \( q \in Q \) that \( \mathbb{E}^q[u] = H(q) \). \( \Box \)

(Several lectures are skipped.)

1.2 Risk Aversion

Risk aversion is a property of most humans, and it means that an agent would refuse a fair gamble. This is captured mathematically by concave utility functions. If \( U \) is concave then (among other things) \( U(0) \geq U(-\frac{1}{2}) + U(\frac{1}{2}) \), so the risk-averse agent prefers no gamble to the gamble with outcomes \(-1\) and \(1\) with equal probability.

For this section let \( A = \mathbb{R} \) and let \( U \) be a strictly increasing, strictly concave utility function (so the agent prefers more to less and is risk-averse). Let \( W_0 \) be the agent's initial wealth. The market consists of \( n \) risky assets with (random) return \( \tilde{r} \) and one riskless asset with known return \( r_f \), over one time period. Let \( a \) be the vector of amounts of the agent's initial wealth invested in the risky
assets. Then the agents (random) final wealth is
\[ \tilde{W} = W_0(1 + r_f) + \sum_{i=1}^{n} (-a_i(1 + r_f) + a_i(1 + \tilde{r}_i)) \]
\[ = W_0(1 + r_f) + \sum_{i=1}^{n} a_i(r_i - r_f) \]
Because the agent is risk-averse, \( a_i > 0 \) only if \( \mathbb{E}[\tilde{r}_i - r_f] > 0 \). The agent’s optimal investment problem is to solve
\[ \max_a \mathbb{E}[U(\tilde{W})] = \max_a \mathbb{E} \left[ U \left( W_0(1 + r_f) + \sum_{i=1}^{n} a_i(r_i - r_f) \right) \right] \]
If \( U \) is continuously differentiable then a necessary and sufficient condition for the optimal investment is \( \mathbb{E}[U'(\tilde{W})(\tilde{r} - r_f)] = 0 \) for all \( i = 1, \ldots, n \). (This is the first-order condition. The second-order condition is automatically satisfied and the maximum is unique because \( U \) is strictly concave.)
Suppose now that \( n = 1 \), so there is one asset. From the first-order condition, the agent invests nothing in the risky asset if and only if \( \mathbb{E}[\tilde{r} - r_f] \leq 0 \), i.e. if and only if \( \mathbb{E}[\tilde{r} - r_f] \leq 0 \). We will require \( \mathbb{E}[\tilde{r} - r_f] > 0 \) so that \( a^* > 0 \). The agent invests all of his initial wealth in the risky asset if and only if \( \mathbb{E}[U'(W_0(1 + \tilde{r}))(\tilde{r} - r_f)] \geq 0 \).
Assuming \( \tilde{r} - r_f \) is small, apply Taylor’s theorem to see
\[ 0 \leq \mathbb{E}[U'(W_0(1 + \tilde{r}))(\tilde{r} - r_f)] \]
\[ \approx \mathbb{E}[U'(W_0(1 + r_f))(\tilde{r} - r_f) + U''(W_0(1 + r_f))(\tilde{r} - r_f)^2W_0 + O(\tilde{r} - r_f)^3] \]
so
\[ \mathbb{E}[\tilde{r} - r_f] \geq \frac{U''(W_0(1 + r_f))}{R_A(W_0(1 + r_f))} \mathbb{E}[(\tilde{r} - r_f)^2] W_0. \]

Arrow and Pratt define the absolute risk-aversion to be \( R_A(z) = -\frac{U''(z)}{U'(z)} \) and the relative risk-aversion to be \( R_R(z) = zR_A(z) \).

1.2.1 Theorem. If \( \frac{dR_A}{dz} < 0, > 0, = 0 \) then \( \frac{da^*}{dW_0} > 0, < 0, = 0 \), respectively.

Proof: See Assignment 1.

1.2.2 Theorem. If \( \frac{dR_R}{dz} < 0, > 0, = 0 \) then \( \frac{d\eta}{dW_0} > 0, < 0, = 0 \), respectively, where \( \eta = \frac{a^*}{W_0} \).
1.2.3 Examples.
(i) $U(x) = -e^{-\lambda x}$, $\lambda > 0$, is exponential utility. $U'(x) = \lambda e^{-\lambda x}$ and $U''(x) = -\lambda^2 e^{-\lambda x}$, so $R_A = \lambda$ is constant, so the exponential utility function has constant absolute risk aversion (CARA).
(ii) $U(x) = x^{1-\gamma}/(1 - \gamma)$, $\gamma \in (0, 1)$ is power utility. $U'(x) = x^{-\gamma}$ and $U''(x) = -\gamma x^{-\gamma-1}$ so $R_R = \gamma$ is constant, so the power utility function has constant relative risk aversion (CRRA).

2 One-period models

2.1 Markets

We suppose that there are $S$ possible states of the world $\omega_1, \ldots, \omega_S$ one period from today, with associated probabilities $p_1, \ldots, p_S$. A market with $N$ securities is specified by an $N \times S$ matrix $D$ and an vector $q \in \mathbb{R}^N$, where $D_{ij}$ is to be interpreted as the payoff of the $i$th security in state $\omega_j$, and $q$ are the prices of the securities today. An arbitrage is a portfolio that makes a risk-free gain. More precisely, an arbitrage is a portfolio $\theta \in \mathbb{R}^N$ such that either $q \cdot \theta < 0$ and $D^T \theta \geq 0$ or $q \cdot \theta < 0$ and $D^T \theta > 0$.

2.1.1 Theorem. There is no arbitrage in the market $(q, D)$ if and only if there is a state price vector, i.e. a $\psi \gg 0$ such that $q = D\psi$.

Proof: Let $M = \{(-q \cdot \theta, D^T \theta) \mid \theta \in \mathbb{R}^N\} \leq \mathbb{R}^{1+S}$ and let $K = \mathbb{R}^1_{+} \times \mathbb{R}^S$, the positive cone in this space. There is no arbitrage if and only if $M \cap K = \{0\}$. If this is the case then by the separating hyperplane theorem for closed convex cones there is a linear functional $F$ such that $F(m) < F(k)$ for all $m \in M$ and $k \in K$, $k \neq 0$. Since $M$ is linear subspace we must have $F(m) = 0$ for all $m \in M$, so $F(k) > 0$ for all $k \in K$, $k \neq 0$. Hence $F(x, y) = \alpha x + \beta \cdot y$ for some $\alpha > 0$ and $\beta \gg 0$, and $0 = \alpha(-q \cdot \theta) + \beta \cdot (D^T \theta)$ for all $\theta \in \mathbb{R}^N$. Whence $\psi := \frac{1}{\beta}\beta$ has the property that $q \cdot \theta = (D\psi) \cdot \theta$ for all $\theta \in \mathbb{R}^N$, so $q = D\psi$. Conversely, if $q = D\psi$ for all $\psi \gg 0$ then

$q \cdot \theta > 0 \iff \psi \cdot D^T \theta > 0 \iff D^T \theta > 0.$

$\psi_1$ may be thought of as the price of a security that pays one unit in state $\omega_1$ and zero in all other states (i.e. the price of an Arrow-Debreu security).

For this section a utility function is a function $U : \mathbb{R}^S \to \mathbb{R}$ that is strictly increasing, strictly concave, and continuously differentiable. The information defining an agent is its initial endowment $e \in \mathbb{R}^S$ and its utility function $U$. The agent’s problem in a given market $(q, D)$ is to solve

$$
\sup_{e \in X(e, q)} U(e) \text{ where } X(e, q) := \mathbb{R}_+^S \cap \{e + D^T \theta \mid \theta \in \mathbb{R}^N, q \cdot \theta \leq 0\}.
$$

Remark. If there is a portfolio $\theta^0$ such that $D^T \theta^0 > 0$ then $q \cdot \theta^* = 0$ at the optimal portfolio $\theta^*$ since utility functions are increasing.
2.1.2 Theorem. There is no arbitrage if and only if the agent’s problem has a solution.

2.1.3 Theorem. Suppose the agent’s problem has a solution $c^* \gg 0$, and further that $\partial U(c^*) \gg 0$. Then the state price vector is a multiple of $\partial U(c^*)$ (the gradient of $U$ at $c^*$).

Proof: Since $c^* \gg 0$, for any $\theta$ there is $\varepsilon > 0$ such that $c^* + \alpha D^T \theta \geq 0$ for all $\alpha \in [-\varepsilon, \varepsilon]$. Suppose $q \cdot \theta = 0$. The function $\alpha \mapsto U(c^* + \alpha D^T \theta)$ is continuously differentiable and maximized at $\alpha = 0$. Therefore $\delta U(c^*) \cdot D^T \theta = 0$. It follows that the subspace of $\mathbb{R}^N$ orthogonal to $q$ is equal to the subspace orthogonal to $D\delta U(c^*)$, so these vectors are scalar multiples. If $\delta U(c^*) \gg 0$ then the appropriate multiple is a state price vector. □

Remark. $\delta U(c^*) \gg 0$ is always satisfied because $U$ is assumed (among other things) to be concave and strictly increasing.

2.1.4 Corollary. If $c^* \gg 0$ and $\partial U(c^*) \gg 0$ then $c^*$ is optimal if and only if $\lambda \partial U(c^*)$ is a state price vector for some $\lambda$.

2.2 Equilibrium

2.2.1 Definition. Let there be $m$ agents $(e^1, U_1), \ldots, (e^m, U_m)$ and a dividends matrix $D$. An equilibrium for this marketplace is $(\theta^1, \ldots, \theta^m, q)$ such that

(i) $c^i := e^i + D^T \theta^i$ solves $\sup_{c \in X(e^i, q)} U_i(c)$ for each $i$; and

(ii) $\sum_{i=1}^m \theta^i = 0$ (the market clearing condition).

Remark. Generally, when solving for prices, solve for $\theta^i$ as a function of $q$ and then determine $q$ via the market clearing condition.

2.2.2 Example. If there is a single agent, $m = 1$, then we must have $c^* = e$ since $\theta^* = 0$ by the market clearing condition. If $e \gg 0$ then $q$ is a multiple of $\partial U(e)$. (The prices in a one agent marketplace are such that the agent would not be interested in trading even if there were someone else with whom to trade – a no-trade equilibrium.)

Since the one agent case is so easy to solve, it would be useful to know when the multiple agent case can be reduced to solving the problem for a single representative agent.

2.2.3 Definition. In a marketplace with agents $(e^1, U_1), \ldots, (e^m, U_m)$, let $e = \sum_{i=1}^m e^i$. A “vector” $(e^1, \ldots, e^m) \in (\mathbb{R}^S)^m$ is said to be a feasible consumption allocation if $\sum_{i=1}^m e^i \leq e$, and a feasible consumption allocation $c$ is Pareto optimal if there does not exist another feasible consumption allocation $\tilde{c}$ such that $U_i(\tilde{c}^i) \geq U_i(c^i)$ for all $i$ and $U_j(\tilde{c}^j) > U_j(c^j)$ for some $j$. 
2.2.4 Examples.
(i) One agent consuming \( e \) (and all others nothing) is Pareto optimal.
(ii) An equilibrium allocation is not necessarily Pareto optimal, however...

2.2.5 Theorem. If the market is complete (i.e. \( \text{colspan}(D) = \mathbb{R}^S \)) then any equilibrium is Pareto optimal.

2.2.6 Definition. In a marketplace with agents \((e_1, U_1), \ldots, (e_m, U_m)\), a representative agent is an agent with endowment \( e = \sum_{i=1}^{m} e^i \) and utility function

\[
U_{\lambda}(x) := \sup_{\sum_{i=1}^{m} c^i \leq x} \sum_{i=1}^{m} \lambda U_i(c^i)
\]

2.2.7 Lemma. If each agent’s utility function is concave then a feasible consumption allocation \( c \) is Pareto optimal if and only if there is \( \lambda > 0 \) such that \( c \) solves \( \sup_{c, \text{feas.}} U_{\lambda}(c) \).

Proof: Let \( U(x) := (U_1(x), \ldots, U_m(x))^T \) and

\[
U := \{ U(x) - U(c) - z \mid x \text{ feasible}, z \in \mathbb{R}^m_+ \},
\]

and let \( K = \mathbb{R}^m_+ \setminus \{0\} \). \( U \) is convex since the \( U_i \) are concave, and \( U \cap K = \{0\} \) since \( c \) is Pareto optimal. By the separating hyperplane theorem there is \( \lambda \) such that \( \lambda \cdot (U(x) - U(c) - z) \leq \lambda \cdot k \) for all elements of \( U \) and \( k \in K \). \( 0 \in U \), so \( \lambda \geq 0 \), and \( \lambda \cdot U(c) \geq \lambda \cdot U(x) \) for all \( x \) feasible. The converse is clear. \( \square \)

2.2.8 Theorem. Suppose each agent’s utility function is concave and the market is complete. If \((\theta^i, q)\) is an equilibrium for the market then there is \( \lambda \) such that \((0, q)\) is a no-trade equilibrium for a representative agent \( U_{\lambda} \), i.e. \( q \) is a multiple of \( \partial U_{\lambda}(e) \).

Proof: Since there is an equilibrium there is no arbitrage and there is a state price vector \( \psi \). Since the market is complete each agent’s problem reduces to

\[
\sup_{c \in \mathbb{R}^S_+} U_i(c) \text{ subject to } \psi \cdot c \leq \psi \cdot e^i.
\]

If an agent’s endowment is zero then they cannot affect the market, so we may assume that \( e^i > 0 \) for all \( i \).

By a theorem from convex optimization (the saddle point theorem), for each \( i \) there is a Lagrange multiplier \( \alpha_i \geq 0 \) such that the optimal consumption allocation \( c^i = e^i + D^T \theta^i \) solves

\[
\sup_{c \in \mathbb{R}^S_+} U_i(c) - \alpha_i(\psi \cdot c - \psi \cdot e^i).
\]
In fact, $\alpha_i > 0$ and the constraint is binding at optimum. Let $\lambda_i = 1/\alpha_i$. For any feasible consumption allocation $(x^1, \ldots, x^m)$,

$$\sum_{i=1}^{m} \lambda_i U_i(c^i) = \sum_{i=1}^{m} \lambda_i U_i(c^i) - \lambda_i \alpha_i (\psi \cdot c^i - \psi \cdot e^i)$$

$$\geq \sum_{i=1}^{m} \lambda_i (U_i(x^i) - \alpha_i (\psi \cdot x^i - \psi \cdot e^i))$$

$$= \sum_{i=1}^{m} \lambda_i U_i(x^i) - \psi \cdot \sum_{i=1}^{m} (x^i - e^i) \geq \sum_{i=1}^{m} \lambda_i U_i(c^i)$$

Therefore $(c^1, \ldots, c^m)$ solves the problem of the representative agent defined by $\lambda$. It remains to show that $(0, q)$ is a no-trade equilibrium for $(e, U_\lambda)$. Suppose that there is $x \in \mathbb{R}^m_+$ with $U_\lambda(x) > U_\lambda(e)$ and $\psi \cdot x \leq \psi \cdot e$. Then there is $(x^1, \ldots, x^m)$ such that $\sum_{i=1}^{m} \lambda_i U_i(x^i) > \sum_{i=1}^{m} \lambda_i U_i(c^i)$. Trivially,

$$\sum_{i=1}^{m} \lambda_i \alpha_i \psi \cdot x^i = \psi \cdot x \leq \psi \cdot e = \sum_{i=1}^{m} \lambda_i \alpha_i \psi \cdot c^i.$$

Whence we have the following contradiction to the saddle point theorem.

$$\sum_{i=1}^{m} \lambda_i (U_i(x^i) - \alpha_i (\psi \cdot x^i - \psi \cdot e^i)) > \sum_{i=1}^{m} \lambda_i (U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i)) \quad \square$$

Remark. In a representative agent market, adding or removing a security in net supply zero would not change the prices of the other assets.

3 Multi-period models

3.1 Utility

There was a confusing lecture on different common choices for multi-period utility functions. We will stick mostly with additive utility $U(c) := \mathbb{E}\left[\sum_{t=0}^{T} u_t(c_t)\right]$.

3.2 Markets

Read Chapter 2, sections A, B, and C, of Duffie now.

3.3 Arbitrage and equivalent martingale measures

Note that $\theta_t \cdot S_t = \mathbb{E}^{Q}\left[\sum_{j=t+1}^{T} \delta_{t,j}^{\theta} \frac{b_{j}}{\nu_{t,j}}\right]$ for every trading strategy $\theta \in \Theta$ if and only if $S_t = \mathbb{E}^{Q}\left[\sum_{j=t+1}^{T} \frac{b_{j}}{\nu_{t,j}}\right]$. We need to find the change-of-measure $\frac{dQ}{dP} = \xi_T$ on
Consider the market \((\delta, S)\) with no arbitrage. Let \(\pi\) be the associated state-price deflator. We say that \(\pi\) is a redundant security if there is \(\theta \in \Theta\) such that \(\delta^\theta_t = \delta_t\) for all \(t > 0\). By the usual no-arbitrage argument we must have
\[
\hat{S}_t = \frac{1}{\pi_t} \mathbb{E}_t[\sum_{j=t+1}^T \pi_j \delta_j].
\]

### 3.4 American securities

Suppose that \((\delta, S)\) is a complete market with no arbitrage. Then \(\pi\) is unique up to a scalar. An American security is a pair consisting of a payoff process and a stopping time, \((X, \tau)\). For stopping times \(\tau \leq \hat{\tau}\), \(\delta^X,\tau = 0\) if \(t \in (0, \tau)\) and \(X_\tau\) if \(t = \tau\). The price of this security should be \(V^*_\tau = \sup_{\tau \in T(0)} \mathbb{E}[X_\tau \pi_\tau]\) where \(T(t) = \{\tau \mid t \leq \tau \leq \hat{\tau}\}\). If the price on the market is less than this value then an arbitrage would be to buy this option and exercise it optimally, so the market price is at least \(V^*_\tau\). On the other hand if the market price is greater than \(V^*_\tau\) then sell the option and create a trading strategy \(\theta\) such that \(V^\theta_t \geq X_t\). How to do this? (Note that it is not an arbitrage to simply put the money in the bank. On average that strategy will make money, but an arbitrage needs to be non-negative almost surely.)

Let \(Y\) be any process. \(W_t = \sup_{\tau \in T(0)} \mathbb{E}[Y_\tau]\) is the Snell envelope of \(Y\). Then \(W\) is a super-martingale, and is a martingale until the optimal exercise time. For our purposes, consider the \(W\) associated with \(Y = \pi X\), and write...
\( W = Z - A \) (the Doob-Meyer decomposition). Let \( \theta \) be such that

\[
\delta^\theta_t = \begin{cases} 0 & \text{if } 0 < t < \hat{\tau} \\ \frac{Z_t}{\pi_t} & \text{if } t = \hat{\tau} \end{cases}
\]

Then \( V^\theta_0 = \frac{Z_0}{\pi_0} = W_0 = V^*_0 \) and \( \pi_t V^\theta_t = \mathbb{E}_t[Z_{\hat{\tau}}] = Z_t \) for all \( t \), so \( V^\theta_t = \frac{Z_t}{\pi_t} = \frac{1}{\pi_t}(W_t + A_t) \geq \frac{1}{\pi_t}(Y_t + A_t) = X_t + \frac{A_t}{\pi_t} \geq X_t \), and the strategy is a super-hedge.

### 3.5 Methods of solving for prices

Suppose we have a market where the \( i^{th} \) agent has utility function

\[
U^i_t(c) = \mathbb{E}_t \left[ \sum_{j=t}^{T} u^i_j(c_j) \right],
\]

where the \( u^i_j \) are strictly increasing, concave, and twice continuously differentiable on \((0, \infty)\). Assume that the equilibrium is Pareto optimal (in particular, this is the case when the market is complete).

(i) Consider a representative agent

\[
U^{\lambda}_t(x) := \mathbb{E}_t \left[ \sum_{j=t}^{T} u^{\lambda}_{\lambda j}(\tilde{e}_j) \right], \quad \text{where } u^{\lambda}_{\lambda j}(x) := \sup_{c^1_j + \ldots + c^m_j \leq x} \sum_{i=1}^{m} \lambda_i u^i_j(c^i_j).
\]

(ii) Find a no-trade equilibrium for \((U^\lambda, e^1 + \ldots + e^m)\), giving a price process \( S_t(\lambda) \) and find optimal arguments \( \tilde{c}^i_j(\lambda) \) by solving the problem for \( u^{\lambda}_{\lambda j} \) at \( x = e_j \).

(iii) Solve each agent’s problem using the price process \( S_t(\lambda) \) for \( c^i(\lambda^*) \). Find \( \lambda^* \) so that \( c^i(\lambda^*) = \tilde{c}^i_j(\lambda^*) \) for all \( i \) and \( j \).

Then \( S_t(\lambda^*) \) are the equilibrium prices and \( c^i(\lambda^*) \) are the equilibrium consumption allocations. Market clearing is automatic.

### 3.6 Dynamic programming in the Markov setting

The third step is often the most difficult. It is often useful to take the dynamic programming approach. Suppose that \( X \) is a time-homogeneous Markov chain with state-space \( Z = \{1, \ldots, k\} \), i.e. \( \mathbb{P}[X_0 = i] = 1 \) and

\[
\mathbb{P}[X_t = j \mid \mathcal{F}^X_{t-1}] = \mathbb{P}[X_t = j \mid X_{t-1}] =: q_{X_{t-1}, j}.
\]

(In particular the transition matrix does not depend on time.) We think of \( X \) as a collection of macroeconomic indicators.

#### 3.6.1 Lemma. For all \( f : Z^{T-t-1} \to \mathbb{R} \), \( \mathbb{E}[f(X_{t+1}, \ldots, X_T) \mid \mathcal{F}^X_t] = g(X_t) \) for some \( g : Z \to \mathbb{R} \).
We study the single agent optimization problem where agent has utility function

$$U_t(c) = \mathbb{E}_t \left[ \sum_{j=t}^{T} u_j(c_j) \right]$$

and endowment $e_t = g_t(X_t)$, and the dividend process is the market is $\delta_t = f_t(X_t)$. Suppose that $(S_t, \delta_t)$ admits no arbitrage and that we can write $S_t = S_t(X_t)$. Then

$$S_t(X_t) = \mathbb{E}_t [u'_{t+1}(g_{t+1}(X_{t+1}))(f_{t+1}(X_{t+1}) + S_{t+1}(X_{t+1}))] / u'_t(g_t(X_t)),$$

the stochastic Euler equation. Let $\mathcal{C} \subseteq \mathbb{L}_+$ be the set of feasible consumption processes and $\Theta$ be the set of feasible trading strategies (adapted processes for which the associated consumption process is in $\mathcal{C}$). We need to find $(c^*, \theta^*) \in \mathcal{C} \times \Theta$ such that $c^*$ solves $\sup_{(c, \theta)} U_0(c)$ and $c^*$ is given by the trading strategy $\theta^*$. Define the value function

$$V_t(i, w) = \sup_{(c, \theta) \in \mathcal{C} \times \Theta} \mathbb{E} \left[ \sum_{j=t}^{T} u_j(c_j) \mid X_t = i \right]$$

subject to $W^0_t = w$

$$c_j + \theta_j S_j(X_j) \leq W^0_j + g_j(X_j) \text{ for all } j = t, \ldots, T$$

where $W^0_t$ is the wealth process associated with $\theta$, defined as

$$W^0_t := \theta_{t-1}(S_t(X_t) + f_t(X_t)).$$

Note that the second condition is the same as requiring $c_j \leq \delta_j + e_j$ for all $j$. To solve this we apply the Bellman principle. Define

$$F_{t+1}(i, w) := 0,$$

$$G_{t,i}(c_t, \theta_t) := u_t(c_t) + \mathbb{E}[F_{t+1}(X_{t+1}, \theta_t(S_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})))) \mid X_t = i]$$

$$F_t(i, w) := \sup_{(c, \theta) \in \mathbb{R}_+ \times \mathbb{R}_+} G_{t,i}(c, \theta) \text{ subject to } c + \theta S_t(i) \leq w + g_t(i)$$

Is $F_t(i, w) = V_t(i, w)$? For any policy $(c, \theta) \in \mathcal{C} \times \Theta$,

$$F_t(X_t, w) = G_{X_t,i}(c, \theta) = u_t(c_t) + \mathbb{E}[F_{t+1}(X_{t+1}, W^0_{t+1})],$$

so $\mathbb{E}[F_t(X_t, W^0_t)] - \mathbb{E}[F_{t+1}(X_{t+1}, W^0_{t+1})] \geq u_t(c_t)$ for all $t$. Summing over those $j = t, \ldots, T$, we see $F_t(X_t, w) \geq \sum_{j=t}^{T} u_j(c^*_j) \mid X_t \geq V_t(X_t, w)$ for all $c$.

Now take the optimal policy $(c^*, \theta^*) \in \mathcal{C} \times \Theta$ for the problem of $F$ (not necessarily optimal for the problem of $V$). Then

$$F_t(X_t, w) = G_{X_t,i}(c^*, \theta^*) = u_t(c^*_t) + \mathbb{E}[F_{t+1}(X_{t+1}, W^0_{t+1})],$$

so $\mathbb{E}[F_t(X_t, W^0_t)] - \mathbb{E}[F_{t+1}(X_{t+1}, W^0_{t+1})] = u_t(c^*_t)$, for all $t$. Summing over $j = t, \ldots, T$, we see $F_t(X_t, w) = \sum_{j=t}^{T} u_j(c^*_j) \mid X_t$, and it must be the case that $F_t(i, w) = V_t(i, w)$, with optimal policy for the problem of $V$ also $(c^*, \theta^*)$.  

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3.6.2 Proposition. If $c + \theta S(t) \leq w + g(t)$ is feasible then the optimization problem for $F_t(i, w)$ has a solution.

3.6.3 Proposition. $F$ is differentiable in $w$ and $\frac{\partial}{\partial w} F_t(i, w) = u_i'(c_t^*)$.

Proof: Write $c = w + g_t(i) - \theta S_t$ and plug it in. (??) □

3.6.4 Proposition. If $c^* \gg 0$ then

$$S_t(X_t) = E_i \left[ u_{t+1}(c_{t+1}^*) (S_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})) \mid X_t \right]$$

As before, for the single agent problem, $c^* = e$.

3.7 Infinite horizon

Again let $X_t$ be a time homogeneous Markov chain on states $Z$, and

$$\delta_t = f(X_t), \quad e_t = g(X_t), \quad U(c) = E \left[ \sum_{t=0}^{\infty} \rho^t u(c_t) \mid X_0 = i \right],$$

where $\rho \in (0, 1)$ and $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, concave, continuously differentiable on $(0, \infty)$, and bounded. (Note that $g, f, \rho, u$ do not depend on $t$). Set $u \equiv -\infty$ on $\mathbb{R}_-$ by convention. Let $\mathcal{L}$ be the set of adapted, bounded processes. For now we will consider only trading strategies bounded below by $-b$ (a limit on short-selling). Let $\Theta$ be the set of these strategies.

Suppose $S_t = S(X_t)$ is a given set of prices. Define

$$V(i, w) = \sup_{(c, \theta) \in \mathcal{L} \times \Theta} E_i \left[ \sum_{t=0}^{\infty} \rho^t u(c_t) \mid X_0 = i \right]$$

subject to $W_{1}^{\theta} = w$ and $c_j + \theta_j S(k) \leq W_{1}^{\theta} + g(k)$ for all $k = 0, 1, 2, \ldots$ and $j \in Z$, where $W_{1}^{\theta} := \theta_{t-1}(S(X_t) + f(X_t))$.

Proceeding heuristically to derive the Bellman equation,

$$V(i, w) = \sup_{(c, \theta) \in \mathcal{L} \times \Theta} u(c_0) + \rho E^i \left[ \sum_{t=0}^{\infty} \rho^t u(c_{t+1}) \right]$$

$$= \sup_{(c, \theta) \in \mathbb{R}_+ \times [-b, \infty)} u(c_0) + \rho E^i [V(X_1, W_1^\theta)].$$

Define a sequence of functions be defining a transformation

$$UF(i, w) = \sup_{(c, \theta)} u(c_0) + \rho E^i [F(X_1, W_1^\theta)]$$

subject to $W_1^\theta = \theta(S(X_1) + f(X_1))$ and $c + \theta S(i) = w + g(i)$
Let \( F^0 \equiv 0 \) and define \( F^{j+1} = UF^j \).

Let \( D = Z \times [w, \infty) \) and say that a function \( F : D \to \mathbb{R} \) is in \( B(D) \) if \( F(i, \cdot) \) is increasing, bounded, and concave. \( B(D) \) is a complete metric space with the supremum metric.

3.7.1 Lemma. If \( F \in B(D) \) then \( UF \in B(D) \), and \( d(UF, UG) \leq \rho d(F, G) \).

Proof: Exercise. \( \square \)

It follows that \( U \) has a (unique) fixed point \( F \), and that function is our candidate for \( V \). Let \( (c(i, w), \theta(i, w)) \) be the optimal consumption in the problem of the fixed point. (These are the feedback functions.) Then define

\[
c^*_t = c(X_t, W_t^*), \quad \theta^*_t = \theta(X_t, W_t^*), \quad W_0^* = w \quad \text{and} \quad W_t^* = \theta^*_{t-1}(S(X_t) + f(X_t))
\]

We claim that \((c^*, \theta^*)\) solves the problem for \( V \) and that \( V = F \). The verification in this case is like the verification in the finite-horizon case. Let \((c_t, \theta_t)\) be any portfolio and \( W^\theta_t = w \), \( W^\theta_t = \theta_{t-1}(S(X_t) + f(X_t)) \). Then \( F(X_t, W^\theta_t) \geq u(c_t) + \rho E_t[F(X_{t+1}, W_{t+1}^\theta)] \). Multiplying by \( \rho \) and summing over \( t = 0, \ldots, T \), we may take the limit (since \( F \) and \( u \) are bounded and \( \rho < 1 \)) to get that \( F(i, w) \geq V(i, w) \). For the constructed optimal portfolio \((c^*, \theta^*)\) we have equalities throughout.

3.7.2 Definition. \( S \) is a Markovian equilibrium if \( c(i, 0) = g(i) \) and \( \theta(i, 0) = 0 \), where \( c \) and \( \theta \) are the feedback functions defined above.

If the endowment is strictly positive then \( S \) is a Markovian equilibrium if and only if

\[
S_t(X_t) = \frac{\rho E_t[u'_{t+1}(g(X_{t+1}))(S_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})) | X_t]}{u'_t(f(X_t))}.
\]

Indeed, \( V \) is strictly concave and strictly increasing. Further, if \( \hat{w} > w \) then \( V(i, \cdot) \) is continuously differentiable at \( w \) and \( \frac{\partial}{\partial w} V(i, \hat{w}) = u'(\hat{c}) \), where \( \hat{c} \) is the optimal consumption at wealth level \( \hat{w} \). To prove the second assertion, let \( v(i, w, X_1) = u(\hat{c} + w - \hat{w}) + \rho V(X_1, W^\theta_1) \), for \( w < \hat{w} \).

\[
E[v(i, w, X_1)] = u(\hat{c} + w - \hat{w}) + \rho E[V(X_1, W^\theta_1)] \leq V(i, w)
\]

since \((\hat{c} + w - \hat{w}) + \hat{\theta} S(i) = w + g(i)\), (i.e. \((\hat{c} + w - \hat{w})\) is a feasible, but not necessarily optimal, consumption in the problem for \( V \)). We have equality above if \( w = \hat{w} \) by definition of \( \hat{c} \). Clearly \( \frac{\partial}{\partial w} v(i, \hat{w}, X_1) = u'(\hat{c}) \), so

\[
 u'(\hat{c}_0)(w - \hat{w}) \leq E[v(i, w, X_1)] - E[v(i, \hat{w}, X_1)] \leq V(i, w) - V(i, \hat{w}) \leq \beta (w - \hat{w})
\]

by concavity for some \( \beta \), and \( \beta \geq u'(\hat{c}_0) \). The argument continues...
3.8 State-price deflator under infinite horizon

Suppose there is no arbitrage. We would like a state-price deflator $\pi$ such that

$$S_t = \mathbb{E}_t[\pi_{t+1}(S_{t+1} + \delta_{t+1})].$$

Consider the finite time-horizon market obtained from the infinite-horizon market by considering those trading strategies for which $\theta_t = 0$ for all $t \geq T + 1$. This market will have no arbitrage, so there will be a state-price deflator $\pi^T$. When possible, define $\tilde{\pi}_t = \pi^T_t$ for some $T > t$. (It is not clear to me why the various $\pi^T$ agree on initial segments.) Unfortunately, a priori $\mathbb{E}[\sum_{t=0}^{\infty} \tilde{\pi}_t \delta^0_t]$ is not necessarily defined. For all bounded stopping times $\tau$,

$$S_{t \wedge \tau} = \frac{1}{\pi_t} \mathbb{E}[\pi_{\tau} S_{\tau} + \sum_{j=t}^{\tau} \pi_j \delta_j \mid \mathcal{F}_t] \mathbf{1}_{\tau \geq t}.$$

Let $\mathcal{L}^*$ denote the collection of adapted processes $x$ for which $\mathbb{E}[\sum_{t=0}^{\infty} |x_t|] < \infty$. We define $\pi$ to be a state price process if it is positive, $\pi \in \mathcal{L}^*$ and $\mathbb{E}[\sum_{t=0}^{\infty} \pi_t \delta^0_t] = 0$ for all $\theta \in \mathcal{L}$.

4 Asymmetric information

4.1 Kyle model (Econometrica, 1985)

We consider two times, $t = 0$ and $t = 1$, and two assets, the risk-less asset with interest rate zero, and the risky asset $v \sim N(\mu_0, \sigma_0)$. There are noisy traders with demand $u \sim N(\theta, \sigma_u)$, informed traders maximizing $\max_p \mathbb{E}[(v - p)x \mid v]$, and market-makers observing $x + u$ and setting $p = \mathbb{E}[v \mid x + u]$. (The market maker is equivalent to having many agents observing $x + u$ and maximizing $\max_p \mathbb{E}[(v - p)x \mid x + u] = \max_p (\mathbb{E}[v \mid x + u] - p)x$. In this case the equilibrium is the rational expectations equilibrium. If the insider believes $p = P(x + u)$ and chooses $x = x^P(v)$ such that $p = \mathbb{E}[v \mid x^P(v) + u]$ then $p = P(x^P(v) + u)$ is the price. Formally, an equilibrium is a pair $(x^*, p^*)$ with $x^* \in \mathcal{F}_0^v$, $p^* \in \mathcal{F}_0^{x^* + u}$ and $\mathbb{E}[(v - p^*)x^* \mid v] \geq \mathbb{E}[(v - p)x \mid v]$ for all $x \in \mathcal{F}_0^v$ (agent optimality) and $p^* = \mathbb{E}[v \mid x^* + u]$ (market efficiency).

From Kyle, assume that $x = \alpha + \beta v$ and $p = \mu + \lambda(x + u)$ are the admissible strategies and admissible pricing rules.

Now generalize to multiple periods $t = 0, 1, \ldots, T$. Assume that the noisy traders total demand by time $t$ is $u_t = \sigma_s B_t$, where $B_{-1} = 0$ and $B$ is a Brownian motion, so that the demand at time $t$ is $u_t - u_{t-1} \sim N(0, \sigma_u)$. Assume the Brownian motion is independent of the demand $v$ of the informed trader. As before, we have the price process $p$ and the total market demand $y_t = x_t + u_t$. The informed trader knows $\mathcal{F}_t^{y_{t-1}}$ at time $t$ and the market maker knows $\mathcal{F}_t^v$ at time $t$. Assume $p_t = \mathbb{E}[v \mid \mathcal{F}_t^v] = p(y_1, \ldots, y_t) = p(X, P)$ and $x_t = x(p_1, \ldots, p_{t-1}, v) = x(X, P)$, and let $\pi_t = \sum_{k=t}^{T} (v - p_k) \Delta x_k = \pi(X, P)$.
be the (random) future profit of the insider. The informed trader aims to maximize $\max_{x, k \geq t} \mathbb{E}[\pi_t \mid \mathcal{F}^{t-1}_{t-1}]$.

Assume for simplicity that $p_t = p_{t-1} + \lambda_t \Delta y_t + h_{t-1}(y_{t-1}, \ldots, y_0)$, where $h$ is linear, and $x_t = x_{t-1} + \beta_t (v - p_{t-1})$.

### 4.1.1 Theorem.
If $p$ has the simplified form above then a linear equilibrium exists, is unique among prices of this form, and $x$ has the simplified form above.

### 4.1.2 Lemma.
If $p^*_t = \mathbb{E}[v \mid \mathcal{F}^{T-1}_{t-1}]$ and has the simplified form above then

$$\mathbb{E}[\pi^*_{t+1} \mid \mathcal{F}^{T-1}_{t}] = \alpha_t (v - p^*_t)^2 + \delta_t.$$ 

**Proof:** At time $T$ there is no profit to be made, so $\alpha_T = \delta_T = 0$. Proceed by induction. $\Box$

### 5 Continuous-time models
It is assumed that the reader understands the basics of derivative pricing when the security price process is specified exogenously. Recall the following technical considerations. If $\theta$ is an adapted stochastic process such that $\mathbb{E}[\int_0^T \theta^2_s ds] < \infty$ for every $T$ then $\int_0^T \theta_s dW_s$ is a martingale. If instead only $\mathbb{P}[\int_0^T \theta^2_s ds < \infty] = 1$ for every $T$ (in this case we say $\theta \in \mathcal{L}^2_{\text{loc}}$) then $\int_0^T \theta_s dW_s$ is a local martingale.

We think of a predictable process $\theta$ as our trading process. Then $\int_0^T \theta_s dS_s$ is the value of our portfolio at time $T$. For this integral to make sense we must have $\sigma \theta \in \mathcal{L}^2_{\text{loc}}$ and $\mu \theta \in \mathcal{L}^1_{\text{loc}}$.

#### 5.1 Black-Scholes model
Let $S_t := S_0 \exp(\alpha t + \sigma W_t)$ be the stock price process, so that its dynamics are given by $dS_t = \mu S_t dt + \sigma S_t dW_t$ (where $\mu = \alpha + \frac{1}{2} \sigma^2$), and $B_t := B_0 e^{rt}$ be the bond price process, so that $dB_t = r B_t dt$. Consider a European call option, which pays $Y_T = (S_T - K)_+$ at maturity $T$. Assume that $Y_t = C(S_t, t)$ for all times $t < T$, for some fixed $C^{2, 1}$ function $C$. We would like to find a self-financing strategy $\theta_t = (\alpha_t, \beta_t)$ that replicates the payoff of the option at maturity. Self-financing means that, for all $t$,

$$\alpha_t S_t + \beta_t B_t = \alpha_0 S_0 + \beta_0 B_0 + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s.$$ 

Technical requirements include that $\alpha \sigma S \in \mathcal{L}^2_{\text{loc}}$ but it is enough to require that $\alpha \in \mathcal{L}^2_{\text{loc}}$ and $\beta \in \mathcal{L}^1_{\text{loc}}$. The derivation is a straightforward application of Itô’s formula. Check that the derived portfolio has $\alpha \in \mathcal{L}^2_{\text{loc}}$. 
The Black-Scholes PDE is \( C_t + \frac{1}{2} \sigma^2 x^2 C_{xx} + r x C_x - r C = 0 \) with boundary condition \( C(x, T) = (x - K)_+ \). According to the Feynman-Kac formula the solution has representation

\[
C(x, t) = \mathbb{E}[e^{-r(T-t)}(Z^x_t - K)_+]
\]

where \( dZ^x_t = rZ^x_u du + \sigma Z^x_u dW_u \) and \( Z^x_t = x \).

In slightly more generality, suppose that stock price process is given by \( dS_t = \mu(S_t, t) dt + \sigma(S_t, t) dW_t \). There are a variety of conditions on \( \mu \) and \( \sigma \) under which the SDE has a unique strong solution. The bond is given by \( dB_t = r(S_t, t) B_t dt \). To price an option with payoff \( Y_T = g(S_T) \), suppose that \( Y_t = V(S_t, t) \) for all times \( t < T \), for some fixed \( C^{2,1} \) function \( V \), so that \( V \) satisfies the PDE

\[
\mathcal{L}_t V - r(x, t)V := V_t + r(x, t)V_x + \frac{1}{2} \sigma^2(x, t)V_{xx} - r(x, t)V = 0
\]

with boundary condition \( V(x, T) = g(x) \).

Suppose that \( \Gamma(x, t, y, s) \) is the transition density for the process defined by \( dZ_t = r(Z_t, t) dt + \sigma(Z_t, t) dW_t \). That is to say, \( \Gamma(x, t, \cdot, s) \) is the conditional density function for the random variable \( Z_s \) given that \( Z_t = x \). \( \Gamma \) is also known as Green’s function for \( \mathcal{L}_t \). When the Green’s function exists

\[
V(x, t) = \mathbb{E}_{x, t}[e^{-\int_t^T r(Z_u) ds} g(Z_T)].
\]

(For some necessary and some sufficient conditions see Friedman’s Parabolic PDE). If \( r, \sigma, r_x, \) and \( \sigma_x \) are bounded and \( \sigma \) is bounded away from zero and \( g \) has polynomial growth then the Green’s function exists.

Keep in mind that there are very strict requirements for the applicability of the Feynman-Kac formula, including that the PDE must have a solution satisfying some growth conditions.

### 5.2 State price deflators

Let \( W \) be a Brownian motion with respect to \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}) \), where the filtration is the augmented natural filtration of \( W \). All processes we are going to mention are to be adapted to this filtration.

Our setup includes a stock price process \( S \) defined by \( dS_t = a_t dt + b_t dW_t \), and we will assume that \( a \in \mathcal{L}^1_{[0, T]}(d\mathbb{P} \times dt) \) and \( b \in \mathcal{L}^2_{[0, T]}(d\mathbb{P} \times dt) \). We will sometimes write \( \mathcal{H}^2 \) for the collection of processes \( b \) such that

\[
\mathbb{E} \left[ \int_0^T \text{tr}(b_t^T b_t) dt \right] < \infty,
\]

i.e. \( \mathcal{H}^2 = \mathcal{L}^2(d\mathbb{P} \times dt) \). Our trading strategies will be predictable processes \( \theta \) such that the portfolio value process \( \int_0^T \theta_u \cdot dS_u \) is well defined, i.e. such that \( b \cdot \theta \in \mathcal{L}^2_{\text{loc}} \) and \( a \cdot \theta \in \mathcal{L}^1_{\text{loc}} \). Assume that there is a bond \( S^0_t := B_t = B_0 \exp \left( \int_0^t r_u du \right) \).
where \( r \in \mathcal{L}^{1}_{\text{loc}} \) is a bounded process.

5.2.1 Definition. An arbitrage in this market is a self-financing trading strategy \( \theta \), i.e. \( \theta_{t} \cdot S_{t} = \theta_{0} \cdot S_{0} + \int_{0}^{t} \theta_{u} \cdot dS_{u} \) for all \( t \in [0, T] \), such that \( \theta_{0} \cdot S_{0} \leq 0 \) and \( \theta_{T} \cdot S_{T} > 0 \) or \( \theta_{0} \cdot S_{0} < 0 \) and \( \theta_{T} \cdot S_{T} \geq 0 \).

A process \( \tilde{\pi} \) is a deflator if it is a strictly positive, adapted, one-dimensional Itô process. A trading strategy \( \pi \) is self-financing with respect to \( S \) if and only if it is self-financing with respect to \( S_{\tilde{\pi}} := \tilde{\pi} \cdot S_{t} \). Indeed,

\[
d(\theta \cdot S_{\tilde{\pi}})_{t} = \theta_{t} \cdot (S_{t}d\tilde{\pi} + \tilde{\pi}dS_{t}) + \tilde{\pi}_{t}S_{t} \cdot d\theta_{t} = \theta_{t} \cdot dS_{\tilde{\pi}} + \tilde{\pi}_{t}S_{t} \cdot d\theta_{t}
\]

and \( \tilde{\pi}_{t}S_{t} \cdot d\theta_{t} = 0 \) if and only if \( \theta \) is self-financing with respect to either (and hence both) of \( S \) and \( S_{\tilde{\pi}} \). It follows that a arbitrage with respect to the original prices is also an arbitrage with respect to the deflated prices, and visa versa. A process \( \pi_{t} \) is a state price deflator if it is a deflator such that \( S_{\pi} \) is a martingale.

5.2.2 Example. Let \( dS_{t} = S_{t}dW_{t} \), \( S_{0} = 1 \), and \( r = 0 \). Let

\[
\tau := \inf \left\{ t \geq 0 \mid \int_{0}^{t} \frac{dW_{s}}{\sqrt{T-s}} = 2 \right\},
\]

a stopping time such that \( \tau < T \) a.s. Then if the portfolio \( \theta = (\alpha, \beta) \), where \( \alpha_{t} = \frac{1}{S_{t}\sqrt{T-t}} \) and \( \beta_{t} \) is determined by the self-financing condition, is admissible then it is an arbitrage, since the level \( S_{\tau} = 2 \) is always reached.

Some possible admissibility conditions include requiring for some \( k \) that \( \mathbb{P}[\theta_{t} \cdot S_{t} \geq k \forall t \in [0, T]] = 1 \) (write \( \theta \in \Theta(S) \)), or requiring that \( \theta \in \mathcal{H}^{2}(S) \).

5.2.3 Theorem. If there is a state price deflator \( \pi \) then there is no arbitrage on \( \Theta(S_{\pi}) \) nor on \( \mathcal{H}^{2}(S_{\pi}) \).

Proof: \( S_{\pi} \) is a martingale, so \( dS_{\pi}^{\pi} = v_{t}dW_{t} \). Then by the self-financing requirement, \( d(\theta_{t} \cdot S_{\pi}) = \theta_{t} \cdot dS_{\pi}^{\pi} = \theta_{t} \cdot v_{t}dW_{t} \), so the portfolio value process is a local martingale. Under either of the conditions it is a super-martingale, so \( \mathbb{E}[\theta_{T} \cdot S_{\pi}^{\pi}] \leq \theta_{0} \cdot S_{0} \). It follows that there is no arbitrage for the deflated prices, and by the remark above there is no arbitrage for the original prices. \( \square \)

5.3 Equivalent martingale measures

Recall that if \( dM_{t} = v_{t}dW_{t} \) and \( \mathbb{E} \left[ \int_{0}^{T} v_{t}^{2}dt \right] < \infty \) then \( M \) is a martingale. Indeed, by the Burkholder-Davis-Gundy inequality, \( \sup_{t \in [0, T]} |M_{t}| \in \mathcal{L}^{1} \), so \( M \) is a martingale.

5.3.1 Definition. Given a \( \mathbb{P} \)-semi-martingale (usually an Itô process) \( X, \mathbb{Q} \) is an equivalent martingale measure (or EMM) for \( X \) if \( \mathbb{Q} \) is equivalent to \( \mathbb{P} \) on \( \mathcal{F}_{T} \) and \( X \) is a martingale under \( \mathbb{Q} \).
We further assume that the process \( \xi_t := \frac{dQ}{dP} |_{F_t} \) has \( \text{Var}(\xi_T) < \infty \).

5.3.2 Theorem. If there is an EMM for \( S \) then there is no arbitrage on \( \Theta(S) \) nor on \( \mathcal{H}_P^2(S) \).

Proof: Let \( \theta \in \mathcal{H}_P^2(S) \) be any trading strategy and let \( dX_t = \theta_t \cdot dS_t \) be the corresponding portfolio value process. Since \( S \) is a \( Q \)-martingale, \( dS_t = v_t dW_t^Q \) and we have

\[
E^Q \sqrt{\xi} = E^P [\sqrt{\xi}] 
\leq \sqrt{E^P \xi} \sqrt{E^P \xi^2} < \infty
\]

Conclude how? \( \blacksquare \)

Remark. The conclusion of the above theorem holds if there is an EMM for \( S^\pi \) for any definor \( \pi \).

5.3.3 Example. Suppose that there is a bond on the market given by \( dB_t = r_t B_t dt \). A natural deflator is \( \tilde{\pi}_t := 1/B_t \). If \( r_t \) is bounded then \( \theta = \tilde{\pi}(S_t^{\tilde{\pi}}) \) if and only if \( \theta \in \Theta(S_t) \) and \( \theta \in \mathcal{H}_P(S_t) \) if and only if \( \theta \in \mathcal{H}_P^2(S_t) \).

Connexion between state price deflator \( \pi \) and EMM density process \( \xi_t \).

5.3.4 Theorem. Suppose there is a bond \( B \) on the market, \( dB_t = r_t B_t dt \) with \( r \) bounded, and \( \tilde{\pi}_t := 1/B_t \).

(i) If \( S \) admits a state price deflator \( \pi \) and \( \text{Var}(\pi_T) < \infty \) then there is an EMM for \( S^\pi \) and \( \xi_t = \frac{\pi_t}{\pi_0} B_t = \frac{\pi_t}{\pi_0} \exp(\int_0^t r_u du) \).

(ii) If \( S^\pi \) admits an EMM there then is a state price deflator \( \pi \) for \( S \) defined by \( \pi_t := \xi_t \tilde{\pi}_t \).

Proof: (i) For any \( s < t \), \( \pi_s S_s = E_s[\pi_s S_t] \) by the definition of state price deflator, so

\[
S_0 = E_s[\frac{\pi_t}{\pi_0} S_t] = E[\xi_t S_t/B_t] = E[\xi_t S_t^\pi]
\]

It follows that \( S^\pi \) is a martingale for the measure defined by \( \frac{dQ}{dP} = \xi_t \).

Fill in the rest of this “obvious” theorem.

(ii) Fill this in. \( \blacksquare \)

Let \( Q \) be any measure equivalent to \( P \). Suppose that \( \xi_t := \frac{dQ}{dP} |_{F_t} \) is a \( P \)-martingale adapted to \( \mathcal{F}^W \). By the martingale representation theorem we can write \( d\xi_t = \gamma_t dW_t \). Let \( \eta = -\frac{\gamma_t}{\xi_t} \), so that \( d\xi_t = -\eta_t \xi_t dW_t \). By Girsanov's theorem, \( dW_t^Q = dW_t + \eta_t dt \) is a \( Q \)-BM. If \( dX_t = \mu_t dt + \sigma_t dW_t \) is an Itô process then \( dX_t = (\mu_t - \sigma_t \eta_t) dt + \sigma_t dW_t^Q \) under \( Q \).

Any \( \eta \) such that \( \mu - \sigma \eta \equiv 0 \) is a market price of risk for \( X \). If the rank of \( \sigma_t \) is not \( N \) (under or over specified) then there might be a problem. If \( \mu - \sigma \eta \equiv 0 \)
Completeness

has no solutions then there is no equivalent martingale measure, and in fact there is an arbitrage. If there is a solution $\eta$ then define

$$
\xi_t = \exp \left( -\int_0^t \eta_u dW_u - \frac{1}{2} \int_0^t \|\eta_u\|^2 du \right)
$$

to obtain a candidate for an EMM for $X$.

Let $\eta$ be a market price of risk for $X$ (so that $\mu - \sigma \eta \equiv 0$). Let $Z = \mathcal{E}(-\eta)$ and assume that $Z$ is a martingale (apply for example Novikov’s criterion). If further $\text{Var}(Z_T) < \infty$ then there is an EMM for $X$, namely $\mathcal{Q}$ defined by $\frac{d\mathcal{Q}}{d\mathcal{P}}|_{\mathcal{F}_T} = Z_T$.

5.4 Completeness

From now on the only admissibility condition that we will consider is that our trading strategies are in $H^2(X)$. For the case of $\Theta(X)$, look up Delbaen and Schachermayer’s NFLVR.

Let $\tilde{\Theta}(X) := \{X_T \cdot \theta_T \mid \theta \text{ self-financing, } \theta \in H^2(X)\}$. Then $\tilde{\Theta}(X) \subseteq L^2(\mathcal{P})$, and the market is complete if the spaces are equal. Note that we need not necessarily be able to replicate the value of contingent claims at times between 0 and $T$. This is slightly different from our definition of complete in the discrete-time case.

5.4.1 Lemma. $\tilde{\Theta}(X)$ is a closed subspace of $L^2(\mathcal{P})$.

5.4.2 Theorem. Suppose $X^{(1)}$ is a bond, so $X^{(1)}_t = \exp(\int_0^t r_u \, du)$, where $r$ is bounded, and there is a bounded market price of risk process for $X$. Then the market is complete if and only if $\text{rank}(\sigma) = d$ for all $t$.

Proof: Define $\mathcal{Q}$ by $\frac{d\mathcal{Q}}{d\mathcal{P}} = \mathcal{E}(-\eta)$. Then the discounted price process $\tilde{X}$ is a $\mathcal{Q}$-martingale. Let $Y \in L^2(\mathcal{P})$ be bounded and let $M_t = \mathbb{E}^\mathcal{Q}[Y \mid \mathcal{F}_t^\mathcal{W}]$, and suppose $dM_t = \varphi_t dW_t^\mathcal{Q}$. If $\sigma$ has full rank then there is $\theta$ such that $\theta_t \sigma_t = \varphi_t$, and $\theta$ will be the replicating portfolio, using the bond to make it self-financing.

Conversely, if $\sigma$ does not have full rank then there is some $\varphi$ such that $\theta \sigma = \varphi$ has no solution $\theta$. The corresponding $Y$ is not replicable. \qed

5.4.3 Definition. Let $Y \in \tilde{\Theta}(X)$ and call $\Psi(Y) := X_0 \theta_0 Y$, where $\theta_0 Y = Y$, $\theta$ is self-financing, and $\theta \in H^2(X)$. An approximate arbitrage is a pair of sequences $Z_n \in \tilde{\Theta}(X)$ with $\Psi(Z_n) \leq 0$ and $Z'_n \in L^2(\mathcal{P})$ with $Z'_n \to Z'$ in $L^2(\mathcal{P})$, such that $Z_n \geq Z'_n$ and $Z' \geq 0$ a.s. and $\mathcal{P}[Z' > 0] > 0$.

Note that an arbitrage is an approximate arbitrage, and in complete markets the notions are the same.

5.4.4 Theorem. Suppose $X^{(1)} \equiv 1$. Then there is no approximate arbitrage if and only if there is an EMM and $\text{Var}(Z_T) < \infty$. 

5.5 Arbitrage pricing with dividends

Let $X$ be the security price process and $D$ be the cumulative dividend process. $G = X + D$ is the gains process. Under an equivalent martingale measure the discounted gains process should be a martingale.

There is some confusion about what it means for a process to be “discounted.”

5.5.1 Definition. A trading strategy $\theta \in L^2_{\text{loc}}(G)$ is self-financing if $\theta_t \cdot (X_t + \Delta D_t) = \theta_0 \cdot G_0 + \int_0^t \theta_s dG_s$. An arbitrage is defined as usual.

6 Term structure

6.1 Heath-Jarrow-Morton

Recall $\lambda_{t,T} = \mathbb{E}_t^Q[e^{-\int_t^T r_u du}]$ is the time $t$ price of a zero-coupon bond that matures at time $T$ and pays one unit.

Model the short rates $f(t,u) = -\frac{d\lambda_{t,T}}{\lambda_{t,T}}$.

Suppose $f(t,u) = f(0,u) + \int_0^t \mu(s,u) ds + \int_0^t \sigma(s,u) dB_s^Q$.

7 Derivative pricing

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with “pricing measure” $\mathbb{Q}$. Suppose that $B$ is a Brownian motion on $[0,T]$ with respect to $\mathbb{Q}$ and the filtration is induced by $B$. Suppose that $r$ is a adapted “spot rate” process such that $\int_0^T |r_s| ds < \infty$. We assume (no reasons given) that $dS_t = r_t S_t dt + \sigma_t S_t dB_t$ when the assets pay no dividends. If the dividends are of the form $dD_t = \delta_t dt$ then $dS_t = (r_t S_t - \delta_t) dt + \sigma_t S_t dB_t$ and

$$S_t = \mathbb{E}_t^Q \left[ e^{-\int_t^T r_u du} S_T + \int_t^T e^{-\int_t^u r_v du} dD_s \right].$$

This last formula also holds if $D$ has a jump part. The time $t$ price of a zero coupon bond maturing at time $s$ is $\Lambda_{t,s} = \mathbb{E}_t^Q[e^{-\int_t^s r_u du}]$.

7.1 Forward contracts

7.2 Futures

The futures price $\varphi$ of claim $W$ is defined so that $\mathbb{E}_t^Q[\int_t^T e^{-\int_t^u r_v du} d\varphi_u] = 0$ for all times $t < T$. By an argument known as “no delivery arbitrage,” we should have $\varphi_T = W$. 
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