Groupoidification in Physics

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Symposium on Category Theory and Physics
PSA Meeting, Montreal
Nov 2010
Program: “Categorify” a quantum mechanical description of states and processes.

We propose to represent:

- configuration spaces of physical systems by **groupoids** (or **stacks**), based on local symmetries
- process relating two systems through time by a **span** of groupoids, including a groupoid of “histories”
We are “doing physics in” the †-monoidal (2-)category \( \text{Span}(\text{Gpd}) \). This relates to more standard picture in \( \text{Hilb} \) by two representations:

- **Degroupoidification** (Baez-Dolan): \( D : \text{Span}_1(\text{Gpd}) \to \text{Hilb} \), explains “Physics in \( \text{Hilb} \)”

- **2-Linearization** (Morton): captures more structure by \( \Lambda : \text{Span}_2(\text{Gpd}) \to 2\text{Hilb} \), suggests “Physics in \( 2\text{Hilb} \).”

Both invariants rely on a **pull-push** process, and some form of **adjointness**.
Definition

A **groupoid** $G$ is a category in which all morphisms are invertible.

Often, we consider groupoids in spaces, manifolds, etc. (i.e. with manifolds of objects, morphisms).

Example

Some relevant groupoids:

- Any set $S$ can be seen as a groupoid with only identity morphisms
- Any group $G$ is a groupoid with one object
- Given a set $S$ with a group-action $G \times S \to S$ yields a transformation groupoid $S//G$ whose objects are elements of $S$; if $g(s) = s'$ then there is a morphism $g_s : s \to s'$
- Any groupoid, as a category, is a union of transformation groupoids (represents “local symmetry”)
A stack is a groupoid taken up to (Morita-)equivalence. This coincides with Morita equivalence for $C^*$ algebras, in the case of groupoid algebras.

Equivalent groupoids are “physically indistinguishable”. (E.g. full action groupoid; skeleton, with quotient space of objects - no need to decide which is “the” stack)

Our proposal is that configuration spaces for physical systems should be (topological, smooth, measured, etc.) stacks.

Note: “configurations” here are roughly “pure states” E.g. energy levels for harmonic oscillator.
A **span** in a category $\mathbf{C}$ is a diagram of the form:

\[
\begin{array}{ccc}
X & \\
\downarrow{s} & \nearrow{t} & \\
A & \quad & B
\end{array}
\]

We’ll use $\mathbf{C} = \mathbf{Gpd}$, so $s$ and $t$ are functors (i.e. also map morphisms, representing symmetries). Spans can be composed by *weak* pullback. (a modified “fibred product”) Span($\mathbf{Gpd}$) gets a monoidal structure from the product in $\mathbf{Gpd}$, and has duals for morphisms and 2-morphisms.
We can look at this two ways:

- **Span** $\mathbf{C}$ is the *universal* 2-category containing $\mathbf{C}$, and for which every morphism has a (two-sided) adjoint. The fact that arrows have adjoints means that Span($\mathbf{C}$) is a $\dagger$-monoidal category (which our representations should preserve).

- Physically, $X$ will represent an object of *histories* leading the system $A$ to the system $B$. Maps $s$ and $t$ pick the starting and terminating *configurations* in $A$ and $B$ for a given history (in the sense internal to $\mathbf{C}$).

(These reasons are closely connected: adjointness is the reversal of time orientation of histories.)
Degroupoidification works like this: To *linearize* a (finite) groupoid, just take the free vector space on its space of isomorphism classes of objects, $\mathbb{C}^A$ (or $L^2(A)$ for more physical situations).

Then there is a pair of linear maps associated to map $f : A \to B$:

- $f^* : \mathbb{C}^B \to \mathbb{C}^A$, with $f^*(g) = g \circ f$ (precomposition)
- $f_* : \mathbb{C}^A \to \mathbb{C}^B$, with $f_*(g)(b) = \sum_{f(a) = b} \frac{\# \text{Aut}(b)}{\# \text{Aut}(a)} g(a)$ (weighted image of functions)

(There are also integral versions; versions with $U(1)$-phased groupoids, etc. for more physical situations)

These are adjoint with respect to a naturally occurring inner product.
Definition

The functor

\[ D : \text{Span}(\text{Gpd}) \to \text{Vect} \]

is defined by

\[ D(G) = C(G) \]

and

\[ D(X, s, t) = t_* \circ s^* \]

This gives multiplication by a matrix counting (with “groupoid cardinality”) the number of histories from \( x \) to \( y \):

\[ D(X)_{([a],[b])} = |(s, t)^{-1}(a, b)|_g \]

This is a “sum over histories”. (For more physics, such as action principle, use \( U(1) \)-groupoids.)
Degroupoidification ignores the fact that $\text{Gpd}$ is a 2-category (with groupoids, functors, and natural transformations). The 2-morphisms of $\text{Span}_2(\text{Gpd})$ are (iso. classes of) spans of span maps:

These have duals, just like the 1-morphisms. We want a representation of $\text{Span}_2(\text{Gpd})$ that captures more than $D$, and preserves the adjointness property for both kinds of morphism.
First, this representation lives in $\mathbf{2Hilb}$:

**Definition**

A finite dimensional **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear abelian category generated by finitely many simple objects. A 2-Hilbert space (Baez) is an abelian $H^*$-category.

That is, 2-vector spaces have a “direct sum” $\oplus$, and $\text{hom}(x, y)$ is a vector space for objects $x$ and $y$. A 2-Hilbert space, in addition, has $\text{hom}(x, y)$ a Hilbert space, and a star structure:

$$\text{hom}(x, y) \cong (\text{hom}(y, x))^*$$

which we think of as finding the “adjoint of a morphism”. A **2-linear map** is a functor preserving all this structure.
Lemma

If $\mathcal{B}$ is an essentially finite groupoid, the representation category $\text{Rep}(\mathcal{B})$ is a 2-Hilbert space.

The “basis elements” (generators) of $[\mathcal{B}, \text{Vect}]$ are labeled by $([b], V)$, where $[b] \in \mathcal{B}$ and $V$ an irreducible rep of $\text{Aut}(b)$.

Baez, Freidel et. al. conjecture the following for the infinite-dimensional case (incompletely understood):

Conjecture

Any 2-Hilbert space is of the following form: $\text{Rep}(\mathcal{A})$, the category of representations of a von Neumann algebra $\mathcal{A}$ on Hilbert spaces. The star structure takes the adjoint of a map.

This includes the example above, by way of the groupoid algebra $C_c(X)$. 
In this context:

- For our physical interpretation $\mathcal{A}$ is the algebras of \textbf{symmetries} of a system. The algebra of \textbf{observables} will be its commutant - which depends on the choice of representation!

- Basis elements are irreducible representations of the vN algebra - physically, these can be interpreted as \textbf{superselection sectors}. Any representation is a direct sum/integral of these.

- Then 2-linear maps are functors, but can also be represented as \textbf{Hilbert bimodules} between algebras. The simple components of these bimodules are like matrix entries.
Definition

A **state** for an object $A$ in a monoidal category is a morphism from the monoidal unit, $\psi : I \rightarrow A$.

- $A \in \text{Hilb}$: state determines a vector by $\psi : \mathbb{C} \rightarrow H$
- $A \in \text{2Hilb}$: a state determines an object (e.g. a representation of groupoid/algebra - an irreducible one is a **superselection sector**)
- $A \in \text{Span}(\text{Gpd})$, the unit is $1$, the terminal groupoid, so

$$
1 \leftarrow S \xrightarrow{\Psi} A
$$

is a “groupoid over $A$”, actually $\Psi$

A state in $\text{Span}(\text{Gpd})$ determines either of the others, using $D$ or $\Lambda$. 
Theorem

If X and B are essentially finite groupoids, a functor f : X → B gives two 2-linear maps:

\[ f^* : \Lambda(B) \to \Lambda(X) \]

namely composition with f, with \( f^* F = F \circ f \) and

\[ f_* : \Lambda(X) \to \Lambda(B) \]

called “pushforward along f”. Furthermore, \( f_* \) is the two-sided adjoint to \( f^* \) (i.e. both left-adjoint and right-adjoint).

In fact, there are left and right adjoints, \( f_* \) and \( f_! \), but the Nakayama isomorphism:

\[ N_{(f,F,b)} : f_!(F)(b) \to f_*(F)(b) \]

is given by the exterior trace map (which uses a modified group average).
Definition

Define the 2-functor $\Lambda$ as follows:

- **Objects:** $\Lambda(B) = \text{Rep}(B) := [B, \text{Vect}]$
- **Morphisms** $\Lambda(X, s, t) = t_\ast \circ s_\ast : \Lambda(a) \to \Lambda(B)$
- **2-Morphisms:** $\Lambda(Y, \sigma, \tau) = \epsilon_{L, \tau} \circ N \circ \eta_{R, \sigma} : (t)_\ast \circ (s)_\ast \to (t')_\ast \circ (s')_\ast$

Picking basis elements $([a], V) \in \Lambda(A)$, and $([b], W) \in \Lambda(B)$, we get that $\Lambda(X, s, t)$ is represented by the matrix with coefficients:

$$\Lambda(X, s, t)([a], V), ([b], W) \simeq \bigoplus_{[x] \in (s,t)^{-1}([a],[b])} \text{hom}_{\text{Rep}(\text{Aut}(x))}(s_\ast(V), t_\ast(W))$$

This is an intertwiner space is the categorified analog of the counting done by $D$: this constructs a Hilbert space as a *direct sum over histories* (generally, direct integral).
In the case where source and target are 1, there is only one basis object in $\Lambda(1)$ (the trivial representation), so the 2-linear maps are represented by a single vector space. Then it turns out:

**Theorem**

Restricting to $\text{hom}_{\text{Span}_2(\text{Gpd})}(1, 1)$:

\[
\begin{array}{c}
A \\
\downarrow s \\
1 \\
\downarrow t \\
X \\
\downarrow \\
1 \\
\downarrow \\
B
\end{array}
\]

where 1 is the (terminal) groupoid with one object and one morphism, $\Lambda$ on 2-morphisms is just the degroupoidification functor $D$.

The groupoid cardinality comes from the modified group average in $N$. 
Example

In the case where $A = B = \text{FinSet}_0$ (equivalently, the symmetric groupoid $\coprod_{n \geq 0} \Sigma_n$ - note no longer finite), we find

$$D(\text{FinSet}_0) = \mathbb{C}[[t]]$$

where $t^n$ marks the basis element for object $[n]$. This gets a canonical inner product and can be treated as the Hilbert space for the \textit{quantum harmonic oscillator} ("Fock Space").

The operators $a = \partial_t$ and $a^\dagger = M_t$, generate the \textit{Weyl algebra} of operators for the QHO. These are given under $D$ by the span $A$:

\[ \begin{array}{c}
\text{FinSet}_0 \\
\cup \ast \\
\downarrow \\
\text{FinSet}_0 \\
\end{array} \quad \begin{array}{c}
\text{FinSet}_0 \\
id \\
\downarrow \\
\text{FinSet}_0 \\
\end{array} \]

and its dual $A^\dagger$. Composites of these give a categorification of operators explicitly in terms of \textit{Feynman diagrams}.
The image of this picture under $\Lambda$ involves representation theory of the symmetric groups as $\Lambda(\text{FinSet}_0) \cong \prod_n \text{Rep}(\Sigma_n)$, and gives rise to “paraparticle statistics”: 

![Diagram with various boxes and arrows representing the representation theory of symmetric groups.](image-url)
Example

An Extended TQFT (ETQFT) is a (weak) monoidal 2-functor

\[ Z : n\text{Cob}_2 \to 2\text{Vect} \]

where \( n\text{Cob}_2 \) is a 2-category of cobordisms.

One construction uses \textit{gauge theory}, for gauge group \( G \) (here a finite group). Given \( M \), the groupoid \( \mathcal{A}_0(M, G) = \text{hom}(\pi_1(M), G)/\!/G \) has:

- **Objects**: Flat connections on \( M \)
- **Morphisms** Gauge transformations

Then \( \mathcal{A}_0(-, G) : n\text{Cob}_2 \to \text{Span}_2(\text{Gpd}) \), and there is an ETQFT
\[ Z_G = \wedge \circ \mathcal{A}_0(-, G). \]
This relies on the fact that cobordisms in $\text{nCob}_2$ can be transformed into products of cospans:

Then $\mathcal{A}_0(-, G)$ maps these into $\text{Span}^2(\text{Gpd})$. 
View $S^1$ as the boundary around a system (e.g. particle).

Irreducible objects of $Z_G(S^1) \simeq [G//G, \textbf{Vect}]$ are labelled by $([g], W)$, for $[g]$ a conjugacy class in $G$ and $W$ an irrep of its stabilizer subgroup.

For $G = SU(2)$, this is an angle $m \in [0, 2\pi]$, a particle; and an irrep of $U(1)$ (or $SU(2)$ for $m = 0$) is labelled by an integer $j$.

This theory then looks like 3D quantum gravity coupled to particles with mass and spin. with mass $m$ and spin $j$.

Under the topology change of the pair of pants, a pair of such reps is taken to one with nontrivial representations (superselection sectors) for all $[mm']$ for any representatives of $[m], [m']$ (each possible total mass and spin for the combined system).

Dynamics (maps between Hilbert spaces) space arises from the 2-morphisms - componentwise in each 2-linear map.