A realizability interpretation for classical arithmetic

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Two flavors of arithmetic

First-order arithmetic comes in two flavors: classical and intuitionistic.

Though the two theories prove the same $\Pi^0_2$ (“computational”) assertions,

- intuitionistic arithmetic has a nice constructive interpretation;

- classical arithmetic does not.
Classical (Peano) arithmetic

Language: $A, \bar{A}, \land, \lor, \forall, \exists$

$\neg \varphi$ is defined using DeMorgan equivalences

Prove sequents $\{\varphi_1, \ldots, \varphi_k\}$

\[
\frac{\Gamma, A, \bar{A}}{\Gamma, \varphi, \Gamma, \bar{A}} \quad \frac{\Gamma, \varphi}{\Gamma, \varphi \lor \psi} \quad \frac{\Gamma, \psi}{\Gamma, \varphi \lor \psi}
\]

\[
\frac{\Gamma, \varphi(x)}{\Gamma, \forall x \varphi(x)} \quad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x \varphi(x)}
\]

\[
\frac{\Gamma, \varphi}{\Gamma, \bar{A} \neg \varphi}
\]

QF axioms \[
\frac{\Gamma, \varphi(0)}{\Gamma, \forall x \varphi(x)}
\] \[
\frac{\Gamma, \neg \varphi(x), \varphi(x')}{\Gamma, \forall x \varphi(x)}
\]
Intuitionistic (Heyting) arithmetic

Language: $\land, \lor, \to, \forall, \exists, \bot$

$\sim \varphi$ is defined as $\varphi \to \bot$

Prove sequents $\{\varphi_1, \ldots, \varphi_k\} \vdash \psi$

\[
\begin{align*}
\Gamma \vdash \varphi & \quad \Gamma \vdash \psi \\
\hline
\Gamma \vdash \varphi \land \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \varphi \land \psi & \quad \Gamma \vdash \varphi \\
\hline
\Gamma \vdash \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma, \varphi \vdash \psi & \\
\hline
\Gamma \vdash \varphi \to \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \varphi \to \psi & \quad \Gamma \vdash \varphi \\
\hline
\Gamma \vdash \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \varphi(0) & \quad \Gamma, \varphi(x) \vdash \varphi(x') \\
\hline
\Gamma \vdash \forall x \varphi(x)
\end{align*}
\]
Normalization vs. cut-elimination

On the intuitionistic side:

- *HA* has a constructive interpretation ("propositions as types," "realizability")
- *HA* comes with a natural set of "simplifying" reductions
- Strong normalization: arbitrary normalization strategies are guaranteed to terminate
- Church-Rosser: various normalization procedures all yield the same result

In contrast, cut-elimination procedures seem less canonical; it is not always clear that the transformations "simplify" the proof.
Maybe the situation isn’t so bad

In an associated paper, I present:

- A realizability interpretation for classical arithmetic
- An new translation of classical arithmetic into intuitionistic arithmetic
- A set of reductions for classical arithmetic

I show:

- Under the translation, my realizability is just intuitionistic realizability plus the Friedman-Dragalin translation
- Under the translation, the reductions are compatible with intuitionistic normalization
- “Typical” finitary and infinitary cut-elimination procedures use the reductions
- With a reasonable restriction, the reductions are strongly normalizing
Conclusions

- It is easy to extract skolem terms from proofs of $\Pi_2$ theorems of classical arithmetic
- Classical arithmetic has a nice set of reductions
- A wide class of cut-elimination procedures all yield the same result
- The Friedman-Dragalin translation is “implicit” in these cut-elimination procedures
The “one-and-a-half negation” translation

Intuitionistically, take \( \sim \varphi \) to be \( \varphi \to \bot \).

Define the following translation from “classical” formulas to “intuitionistic” ones:

\[
\begin{align*}
A^M &= A \\
\overline{A}^M &= \sim A \\
(\varphi \lor \psi)^M &= \varphi^M \lor \psi^M \\
(\varphi \land \psi)^M &= \sim (\sim \varphi \lor \sim \psi)^M \\
(\exists x \varphi)^M &= \exists x \varphi^M \\
(\forall x \varphi)^M &= \sim (\exists x \sim \varphi)^M.
\end{align*}
\]

**Theorem.** Intuitionistically, we have \( \sim \varphi^M \equiv \sim \varphi^N \).

**Corollary.** If \( \{\varphi_1, \ldots, \varphi_k\} \) is provable classically, then

\( (\sim \varphi_1)^M, \ldots, (\sim \varphi_k)^M \vdash \bot \)

intuitionistically (in fact, in minimal logic).

The theorem and corollary still hold true if we define

\( (\varphi \land \psi)^M \equiv \varphi^M \land \psi^M. \)
Translating proofs

Cut,

\[ \Gamma, \varphi \quad \Gamma, \neg \varphi \]

translates to

\[ \vdash \neg \Gamma \quad \vdash \neg \varphi \]

\[ \vdash \neg \Gamma, \neg \varphi \]

The \( \land \) rule,

\[ \Gamma, \varphi \quad \Gamma, \psi \]

translates to

\[ \vdash \neg \Gamma, \neg \varphi \quad \vdash \neg \Gamma, \neg \psi \]

\[ \vdash \neg \Gamma, \neg \varphi, \neg \psi \]
The ∨ rule,

\[
\frac{\Gamma, \varphi}{\Gamma, \varphi \lor \psi}
\]

translates to

\[
\frac{(-\Gamma)^M, (\neg \varphi)^M \vdash \bot}{(-\Gamma)^M \vdash \sim (\neg \varphi)^M \quad \sim (\varphi^M \lor \psi^M) \vdash \sim \varphi^M}
\]

\[
\frac{(-\Gamma)^M, \sim (\varphi^M \lor \psi^M) \vdash \bot}{\bot}
\]
Applying the Friedman-Dragalin translation

Given a proof of $\exists x \ A(x)$ in classical arithmetic, obtain a proof of $\bot$ from $\forall x \sim A(x)$ in arithmetic over minimal logic.

Now, replace $\bot$ everywhere by $\exists x \ A(x)$. This yields a proof of $\exists x \ A(x)$ from

$$\forall x \ (A(x) \rightarrow \exists x \ A(x)),$$

and hence a proof of $\exists x \ A(x)$.

**Corollary.** If classical arithmetic proves $\forall y \ \exists x \ A(x,y)$ then intuitionistic arithmetic proves it as well.
Some reductions

A principal cut:

\[
\begin{array}{c}
d_0 \\
\Gamma, \varphi \lor \psi, \varphi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
d_1 \\
\Gamma, \varphi \lor \psi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
d_1 \\
\Gamma, \neg \varphi \land \neg \psi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\Gamma \\
\hline
\end{array}
\]

reduces to

\[
\begin{array}{c}
d_0 \\
\Gamma, \varphi \lor \psi, \varphi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
d_1 \\
\Gamma, \neg \varphi \land \neg \psi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
d_1 \\
\Gamma, \neg \varphi \land \neg \psi \\
\hline
\end{array}
\]

(invert)

\[
\begin{array}{c}
\Gamma, \varphi \\
\hline
\end{array}
\]

A principal inversion:

\[
\begin{array}{c}
d_0 \\
\Gamma, \varphi \land \psi, \varphi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
d_1 \\
\Gamma, \varphi \land \psi, \psi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
d_1 \\
\Gamma, \neg \varphi \land \neg \psi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\Gamma, \varphi \land \psi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\Gamma, \varphi \\
\hline
\end{array}
\]

reduces to

\[
\begin{array}{c}
d_0 \\
\Gamma, \varphi \land \psi, \varphi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\Gamma, \varphi \\
\hline
\end{array}
\]
A taxonomy of reductions

Add inversion rules: \[ \frac{\Gamma, \varphi \land \psi}{\Gamma, \varphi}, \frac{\Gamma, \forall x \varphi(x)}{\Gamma, \varphi(n)}, \ldots \]

Five kinds of reductions:

1. principal inversions
2. nonprincipal inversion
3. principal cut
4. nonprincipal cut
5. unnecessary free variables
The results

• These reductions are compatible with the normalization of the corresponding intuitionistic proof

• They be used in a Gentzen-style finitary cut elimination procedure

• They are also implicit in infinitary cut elimination procedures

• The Friedman-Dragalin translation corresponds to extracting a witness from a cut-free proof

• The witness extracted is independent of the order in which reductions are applied

• You can eliminate cuts from proofs of $\Sigma_1$ sentences, even without “permutative” reductions

• (Buchholz) If you restrict the permutative reductions, you have strong normalization
Comments

1. Gentzen’s original cut-elimination procedure used a more symmetric cut reduction:

\[
\begin{array}{c}
\frac{d_0}{\Gamma, \forall x \varphi(x), \varphi(y)} \\
\frac{d_1}{\Gamma, \exists x \neg \varphi(x), \neg \varphi(t)}
\end{array}
\]

\[
\frac{\Gamma, \forall x \varphi(x)}{\Gamma}
\]

reduces to

\[
\begin{array}{c}
\frac{d_0}{\Gamma, \forall x \varphi, \varphi(y)} \\
\frac{d_1}{\Gamma, \exists x \neg \varphi, \neg \varphi(t)}
\end{array}
\]

\[
\frac{d_0[t/y]}{\Gamma, \forall x \varphi, \varphi(t)}
\]

\[
\frac{d_1}{\Gamma, \exists x \neg \varphi, \neg \varphi(t)}
\]

\[
\frac{\Gamma, \neg \varphi(t)}{\Gamma}
\]

These are not compatible with normalization, under the translation above.

2. The translation isn’t sharp on fragments of arithmetic; for example, \(I \Sigma_1\) doesn’t translate to \(I \Sigma_1^i\). For one that is (due to Coquand), see

Interpreting classical theories in constructive ones

on my home page.