Semantic approaches to ordinal analysis

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Overview

Ordinal analysis typically proceeds by “unwinding proofs.”

Can we use ordinals, instead, to “build models”?

Motivation:

• Use ideas and methods from model theory, set theory, recursion theory

• Constructions may suggest combinatorial independences
Semantic approaches

- Hilbert and Ackermann: epsilon substitution
- Friedman: models of $\Sigma^1_1$-AC and $ATR_0$
- Paris-Kirby, Sommer, Avigad: $\alpha$-large intervals
- Kripke, Quinsey: fulfillment
- Carlson: ranked partial structures

The $\alpha$-large approach:
- Use ordinals to define large intervals in $\mathbb{N}$
- Carve out models from those

This two-step process becomes difficult for stronger theories.
Another approach

To analyze a theory $T$:

- Use Skolem functions to embed $T$ in a universal theory
- Herbrand’s theorem: it suffices to assign values to finitely many terms, consistent with axioms
- Use ordinals to do this
- Gradually eliminate nonconstructive principles

Advantage: seems to be as flexible as cut elimination

Disadvantage: starts to look less like model theory, and more like cut elimination
Ordinal recursive functions

Fix a system of ordinal notations.

A \( \prec \alpha \)-iterative algorithm is given by a notation \( \beta \prec \alpha \) and elementary functions

- \( \text{start}(\vec{x}) \)
- \( \text{next}(q) \)
- \( \text{norm}(q) \)
- \( \text{result}(q) \)

These data define a function \( F(\vec{x}) \):

\[
\begin{align*}
\text{clock} & \leftarrow \beta \\
\text{state} & \leftarrow \text{start}(\vec{x}) \\
\text{while} \ \text{norm}(\text{state}) \prec \text{clock} \ \text{do} \\
& \quad \text{clock} \leftarrow \text{norm}(\text{state}) \\
& \quad \text{state} \leftarrow \text{next}(\text{state}) \\
\text{return} \ \text{result}(\text{state})
\end{align*}
\]
Ordinal recursive functionals

The previous definition relativizes well.

A relativized $\prec_\alpha$-iterative algorithm is given by a notation $\beta \prec \alpha$ and elementary functions

- $\text{start}(\vec{x})$
- $\text{query}(q)$
- $\text{next}(q, u)$
- $\text{norm}(q)$
- $\text{result}(q)$

These data define a functional $F(\vec{x}, f)$:

\[
\begin{align*}
\text{clock} & \leftarrow \beta \\
\text{state} & \leftarrow \text{start}(\vec{x}) \\
\text{while } \text{norm(state)} & \prec \text{clock} \text{ do} \\
\text{clock} & \leftarrow \text{norm(state)} \\
\text{state} & \leftarrow \text{next}(\text{state}, f(\text{query(state)})) \\
\text{return } \text{result(state)}
\end{align*}
\]
The ordinal analysis of arithmetic

**Theorem.** Suppose $PA(f)$ proves $\forall x \exists y \varphi(x, y, f)$ for some $\Delta_0$ formula $\varphi$. Then there is a $\prec \varepsilon_0$-recursive functional $F(x, f)$ such that $PRA$ proves

$$\forall x, y \ (F(x, f) \downarrow = y \rightarrow \varphi(x, y, f)).$$

This is essentially due to Gentzen, and implies all the usual results of an ordinal analysis.

In the new approach, use “least element” functions to make Peano arithmetic quantifier free:

$$f(x, \vec{z}) = 0 \rightarrow f(\mu_f(\vec{z}), \vec{z}) = 0 \land \mu_f(\vec{z}) \leq x.$$  

Nesting corresponds to complexity of induction.

Goal: given a finite set of $\mu$ axioms, assign consistent values to $\mu$ terms.
The general idea

Suppose $F(x, \mu_0, \mu_1, \ldots, \mu_n)$ is $\prec \alpha$-recursive, and each $\mu_i$ has depth $i$.

Replace this by a $\prec \omega^\alpha$-recursive function $G(x, \mu_0, \ldots, \mu_{n-1})$ which simultaneously computes $F$ and a finite approximation to $\mu_n$ that is consistent with the values used in the computation.

Argument has the flavor of a finite injury priority argument. Start with $\mu_n = \emptyset$. Then:

1. Carry out computation of $F$.

2. If you find a value inconsistent with axiom for the $\mu_n$, correct this value, and repeat.

Assign ordinals to computations, so that the ordinal drops with each step.
The Howard-Bachman ordinal

Let \( \Omega \) denote the first uncountable cardinal, and let \( \varepsilon_{\Omega+1} \) denote the \( \Omega + 1 \)st \( \varepsilon \)-number, i.e. the limit of the sequence

\[
\Omega, \Omega^\Omega, \Omega^{(\Omega^\Omega)}, \ldots
\]

Any ordinal \( \alpha < \varepsilon_{\Omega+1} \) can be written in Cantor normal form to the base \( \Omega \),

\[
\alpha = \Omega^{\alpha_1} \beta_1 + \ldots \Omega^{\alpha_k} \beta_k
\]

where

- \( \alpha > \alpha_1 > \ldots > \alpha_k \)
- each \( \beta_k \) is an element of \( \Omega \).

The \( \beta \)'s occurring in the expansion (as well as in those of the \( \alpha_i \)) are called the components of \( \alpha \).
The Howard-Bachman ordinal (cont’d)

For $\alpha \leq \varepsilon_{\Omega+1}$, define

- $C_\alpha : \Omega \to P(\Omega)$
- $\theta_\alpha : \Omega \to \Omega$

by transfinite recursion, as follows:

\[
C_\alpha(\beta) = \text{the closure of } \{0, 1\} \cup \beta \text{ under } + \text{ and the functions } \theta_\gamma, \text{ where } \gamma < \alpha \text{ and the components of } \gamma \text{ are in } C_\alpha(\beta)
\]

\[
\theta_\alpha = \text{the enumerating function of } \{\delta \mid \delta \notin C_\alpha(\delta) \land \alpha \in C_\alpha(\delta)\}.
\]

One has $\theta_\alpha(\beta) < \theta_\gamma(\delta)$ if and only if one of the following holds:

- $\alpha < \gamma$, $\beta < \theta_\gamma(\delta)$, and all the components of $\alpha$ are less than $\theta_\gamma(\delta)$
- $\alpha = \gamma$ and $\beta < \delta$
- $\gamma \leq \alpha$ but either $\delta$ or some component of $\gamma$ is greater than or equal to $\theta_\alpha(\beta)$.

The Howard-Bachmann ordinal is $\theta_{\varepsilon_{\Omega+1}}(0)$. 
Admissible set theory

The axioms of $KP\omega$ are as follows:

1. Extensionality: $x = y \rightarrow (x \in w \rightarrow y \in w)$
2. Pair: $\exists x (x = \{y, z\})$
3. Union: $\exists x (x = \bigcup y)$
4. $\Delta_0$ separation: $\exists x \forall z (z \in x \leftrightarrow z \in y \land \varphi(z))$ where $\varphi$ is $\Delta_0$ and $x$ does not occur in $\varphi$
5. $\Delta_0$ collection:
   $\forall x \in z \exists y \varphi(x, y) \rightarrow \exists w \forall x \in z \exists y \in w \varphi(x, y)$, where $\varphi$ is $\Delta_0$
6. Foundation: $\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$, for arbitrary $\varphi$
7. Infinity: $\exists x (\emptyset \in x \land \forall y \in x (y \cup \{y\} \in x))$

In the absence of infinity, this is inter-interpretable with $PA$.

**Theorem 0.1** Suppose $KP\omega$ proves $\forall x \exists y \varphi(x, y)$, where $\varphi$ is $\Sigma_1$. Then there is an ordinal $\alpha < \varepsilon_{\Omega+1}$ such that for every $\beta$, we have $\forall x \in L_\beta \exists y \in L_{\theta_\alpha(\beta)} \varphi(x, y)$. 
Primitive recursive set functions

To (re)obtain this result, let us first lift the definition of \(\prec\alpha\)-recursion to functions on sets.

In analogy to the elementary functions on the natural numbers, we need a collection of set functions that is robust, but does not grow too fast.

Use the primitive recursive set functions arising from work of Takeuti, Kino, Jensen, Karp, and Gandy.

Let \(\varphi_\omega (= \theta_\omega)\) be the \(\omega\)th Veblen function.

**Lemma 0.2** For each \(\alpha\), \(L_{\varphi_\omega(\alpha)}\) is closed under the primitive recursive set functions.
Recursion on notations

Now think of $\Omega$ as the order type of the universe. We can define notations for $\varepsilon_{\Omega+1}$ in the class of sets, just as we can define notations for $\varepsilon_0$ in $\mathbb{N}$:

$$\hat{\alpha} = \Omega^{\hat{\alpha}_1} \beta_1 + \ldots \Omega^{\hat{\alpha}_k} \beta_k$$

where $\hat{\alpha}_1, \ldots, \hat{\alpha}_k$ are notations, and $\beta_1, \ldots, \beta_k$ are ordinals.

A $\prec_{\varepsilon_{\Omega+1}}$-recursive functional $F(\bar{x}, f)$ is given by a notation $\hat{\beta} \prec_{\varepsilon_{\Omega+1}}$ and primitive recursive set functions

- $\text{start}(\bar{x})$
- $\text{query}(q)$
- $\text{next}(q, u)$
- $\text{norm}(q)$
- $\text{result}(q)$
Lifting Gentzen’s result

Let $PRS\omega$ be an axiomatization of the primitive recursive set functions (with $\omega$ as a constant).

**Theorem 0.3** Suppose

$$PRS\omega + (\text{Foundation}) \vdash \forall x \exists y \varphi(x, y, \vec{f}),$$

where $\varphi$ is quantifier-free. Then there is a $\prec \varepsilon_{\Omega+1}$-recursive set function $F(x, \vec{f})$ such that

$$PRS\omega \vdash \forall x, y (F(x, \vec{f}) \downarrow = y \rightarrow \varphi(x, y, \vec{f})).$$

Compare to Gentzen’s result for $PA$:

- Foundation replaces induction
- $\varepsilon_{\Omega+1}$ replaces $\varepsilon_0$

We have not said anything about collection yet.
Remember that an instance of $\Delta_0$ collection is of the form

$$\forall v, z (\forall x \in v \exists y \theta(x, y, z) \rightarrow \exists w \forall x \in v \exists y \in w \theta(x, y, z))$$

Rewrite this as

$$\forall v, z (\exists x (x \in v \land \forall y \neg \theta(x, y, z)) \lor \exists w \forall x \in w \exists y \in v \theta(x, y, z)).$$

Pair $v$ and $z$, bring quantifiers to the front, and Skolemize:

$$\forall u, y ((\text{coll}(u) \in (u)_0 \land \neg \theta(\text{coll}(u), y, (u)_1)) \lor \forall x \in u \exists y \in \text{coll}(u) \theta(x, y, (u)_1)).$$

In short, $\text{coll}(\langle v, z \rangle)$ is supposed to return either

- a value $x$ satisfying $x \in v \land \neg \theta(x, y, z)$, or
- a value $w$ satisfying $\forall x \in u \exists x \in w \theta(x, y, z)$. 
Skolemizing collection

Let \( \text{Coll}'(u, y, c) \) denote the primitive recursive relation
\[
(c \in (u)_0 \land \neg \theta((u)_0, y, (u)_1)) \lor \forall x \in u \exists y \in c \ \theta(x, y, (u)_1).
\]
This says “\( c \) is a sound interpretation of \( \text{coll}(u) \) at \( y \).”

Collection is then equivalent to the universal axiom
\[
\forall u, y \ \text{Coll}'(u, y, \text{coll}(u)) \quad (\text{Coll})
\]

\( KP_\omega \) is contained in \( PRS_\omega + (\text{Coll}) + \text{Foundation} \).

**Lemma 0.4** Suppose \( PRS_\omega + (\text{Coll}) + \text{Foundation} \) proves
\[
\forall x \exists y \ \varphi(x, y),
\]
where \( \varphi \) is \( \Delta_0 \). Then there is a \( \prec \varepsilon_{\Omega+1} \)-recursive functional \( F \) such that \( PRS_\omega \) proves
\[
\forall x, y \ (F(x, \text{coll}) \Downarrow y \land \text{Coll}'((y)_0, (y)_1, \text{coll}((y)_0)) \to \varphi(x, y)).
\]

To finish it off, we only need to show that for some \( \alpha \prec \varepsilon_{\Omega+1} \), whenever \( x \) is in \( L_\gamma \), there is an approximation to the \( \text{coll} \) function and a computation of \( F \) in \( L_{\theta_\alpha(\gamma)} \) robust enough to answer the queries and satisfy the final test.
A combinatorial lemma

Lemma 0.5 Suppose $F(x, f)$ is $\hat{\alpha}$-recursive, and $x \in L_\gamma$. Then there is a pair $\langle s, m \rangle \in L_{\theta_\omega+\hat{\alpha}(\gamma)}$ such that

- $m$ is a function,
- $s$ is a computation sequence for $F$ at $x$, $m$, and
- if the result of $s$ is $y$, then $Coll'((y)_0, (y)_1, m((y)_0))$.

Proof: use transfinite induction on $\theta_\omega+\hat{\alpha}(\gamma)$ and a slightly stronger induction hypothesis.

This is analogous to a proof-theoretic “collapsing” lemma.
Conclusion

References:

• “Ordinal analysis without proofs”: from fragments of arithmetic to predicative analysis
• “An ordinal analysis of admissible set theory using recursion on ordinal notations”: admissible set theory
• “Update procedures and the 1-consistency of arithmetic”: a more combinatorial packaging of the ordinal analysis of arithmetic

Further work:

• Rewrite old results: Cut elimination arguments can probably be translated to the new framework. Is there any advantage to doing so?
• Polish the methods: Can one make them seem even more combinatorial, more semantic, and easier to understand?
• Prove new results: Can one use the methods to extract interesting combinatorial principles for ordinals, sets, and numbers?