Sample Solution for HW 2

Constructive Logic

Oct. 05, 2001

**Problem 1** Use truth tables to decide whether the following arguments are classically valid:

(a) Bush will resign or America will go to war if there is a catastrophe. America is going to war with Bush as president. There was a catastrophe.

**Proof.** We can denote the following propositions as follows:

“Bush will resign” $\rightarrow B$ (we will interpret “Bush is a president” and “Bush will NOT resign” ($\neg B$)

“America will go to war” $\rightarrow A$

“There is a catastrophe” $\rightarrow C$

Then the argument can be expressed as:

$$C \rightarrow (B \lor A)$$

$$\begin{array}{c|c|c|c|c|}
A & B & C & C \rightarrow (B \lor A) & A \land \neg B & C \\
T & T & T & T & F & T \\
T & T & F & T & F & F \\
T & F & T & T & T & T \\
T & F & F & T & T & F \\
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F & F & F & T & F & F \\
\end{array}$$

The the bold row of the following truth table shows that the argument is *invalid*.

(b) Bush will resign or America will go to war if there is a catastrophe. If there is a catastrophe while Bush is president, then America will go to war.

**Proof.** We can denote the following propositions as follows:
“Bush will resign” — \( B \) (we will interpret “Bush is a president” and “Bush will NOT resign” \( \neg B \))

“America will go to war” — \( A \)

“There is a catastrophe” — \( C \)

Then the argument can be expressed as:

\[
\frac{C \rightarrow (B \lor A)}{\left( C \land \neg B \right) \rightarrow A}
\]

The bold rows of the following truth table shows that the argument is **valid**.

<table>
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<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>( C \rightarrow (B \lor A) )</th>
<th>( (C \land \neg B) \rightarrow A )</th>
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**Problem 2** Use Kripke models to prove the following:

(a) \( \models \neg \neg (A \lor \neg A) \)

(b) \( \neg \neg A \Rightarrow B \models A \Rightarrow \neg B \)

**Proof.** (a) We want to show that for every Kripke model \( K \), \( K \models \neg \neg (A \lor \neg A) \). This is an abbreviated way of saying that \( \forall i \in K \ i \models \neg \neg (A \lor \neg A) \).

Fix \( K \) and let \( i \) be any world of \( K \). We want to show that \( i \models \neg \neg (A \lor \neg A) \). By the definition of forcing, this is equivalent to saying that \( \forall j \geq i \), \( j \not\models \neg \neg (A \lor \neg A) \), which is equivalent to saying that \( \exists k \geq j \geq i \), \( k \models A \lor \neg A \). In other words, to show that \( i \models \neg \neg (A \lor \neg A) \) it suffices to show that for \( \forall j \geq j \) we can find a \( k \geq j \) such that \( k \models A \lor \neg A \). Now let us prove that.

Fix arbitrary \( j \geq i \). Now, either there exit a world \( l \geq j \) such that \( l \models A \) or there is none. (Note that here we use a classical argument — law of excluded middle — to reason about what is true at a world of the model.)

**Case 1:** There is such \( l \models A \). Then, let \( k = l \), and we have that \( k \models A \), which implies that \( k \models A \lor \neg A \) by the definition of forcing.

**Case 2:** There is no such \( l \). Then \( j \models \neg A \), so of \( k = j \), \( k \models \neg A \), which
again implies that \( k \models A \lor \neg A \) by the definition of forcing.

Therefore, we can always find \( k \geq j \) such that \( k \models A \lor \neg A \). This completes the proof. \( \blacksquare \)

**Proof.** (b) We want to show that of \( K \) is a model such that \( K \models \neg A \Rightarrow B \), then also \( K \models A \Rightarrow \neg B \).

Fix \( K \models \neg A \Rightarrow B \). We want to show that \( K \models A \Rightarrow \neg B \), i.e. \( \forall i \in K, i \models A \Rightarrow \neg B \). The statement \( i \models A \Rightarrow \neg B \) is equivalent to statement
\( \forall j \geq i, j \models A \Rightarrow \neg B \). As above, the statement \( j \models \neg B \) is equivalent to the statement \( \forall k \geq j \exists l \geq k, l \models B \). Therefore, it suffices to show for \( j \in K \), if \( j \models A \) then \( \forall k \geq j \exists l \geq k, l \models B \).

Fix \( j \models A \) (if no such \( j \) exist we are trivially done). We claim that \( j \models \neg A \). Indeed, by monotonicity \( \forall n \geq j, n \models A \), but than \( \forall n \geq j \exists m \geq n \) (e.g., \( n \) itself) such that \( m \models A \). As we saw in the previous proof this is equivalent to \( j \models \neg A \).

Now we use that fact that \( K \models \neg A \Rightarrow B \), and conclude that \( j \models B \), by the definition of forcing. By monotonicity, we conclude that \( \forall k \geq j \exists l \geq k, l \models B \)

as desired. This concludes that proof. \( \blacksquare \)

**Problem 3** Show that the following sequent is not derivable in constructive logic:

\[ A \Rightarrow B \vdash \neg A \lor B \]

**Proof.** The Soundness Theorem for Kripke semantics states that \( A \Rightarrow B \vdash \neg A \lor B \) implies \( A \Rightarrow B \models \neg A \lor B \). The counter-positive of this statement is \( A \Rightarrow B \not\models \neg A \lor B \) implies \( A \Rightarrow B \not\models \neg A \lor B \). Thus, to show that the sequent \( A \Rightarrow B \vdash \neg A \lor B \) is not derivable it suffices to show that there exist a Kripke model \( K \) such that \( K \not\models \neg A \lor B \) but \( K \models A \Rightarrow B \). The following model has that property:

\[
\begin{array}{c|c}
\sigma_2 & A, B \\
\hline
\sigma_1 & \emptyset
\end{array}
\]

\( \sigma_1 \not\models \neg A \) and \( 1 \not\models B \) so \( 1 \not\models \neg A \lor B \). However, \( 1, 2 \models A \Rightarrow B \). \( \blacksquare \)