Objectives

- Recap of trading strategies
- Explain *why* mean-variance makes sense
- *compute* mean and standard deviation of portfolios
  - one risky asset and risk less
  - multiple risky assets
- Begin discussion of how to implement

Trading strategies

- Basic point: dynamic trading allows you to generate interesting payoff patterns
Which assets

- Portfolio insurance strategy:
  - example so far, stocks and risk free bonds
  - what about
    - domestic and international stocks
    - short term and long term bonds
    - market portfolio and sector bets
    - etc.

Mean-Variance Analysis Why?

- over 90% variation in performance measured by policy weights
- Investor preferences
  - like high expected returns
  - hate variance of portfolio
  - don’t care about anything else
  - single period objective
Formulas

\[ E(\bar{r}) = \sum_{i=1}^{s} i \times \Pr(\bar{r} = i) \]

\[ \sigma_{\bar{r}}^2 = Var(\bar{r}) = E[(\bar{r} - E[\bar{r}])^2] = \sum_{i=1}^{s} (i - E[\bar{r}])^2 \times \Pr(\bar{r} = i) \]

\[ \sigma_{x,y} = Cov(\bar{x}, \bar{y}) = E[(\bar{x} - E[\bar{x}])[\bar{y} - E[\bar{y}])] \]

When does this make sense?

• Normal distributions of returns
  – only mean and variance determine shape of distribution

• Care about deviations from the mean equally...
When doesn’t it make sense?

- Skewed return distributions
- Dynamics Issues and dynamic trading...

Normal vs. Skewed Distribution
Portfolio

- Invest $\omega_i$ in asset $I$
- Total of $N$ assets
- Mean and variance of portfolio?

\[
E[\tilde{r}_p] = \omega_1 E[\tilde{r}_1] + \omega_2 E[\tilde{r}_2] + \cdots + \omega_N E[\tilde{r}_N]
\]

\[
\sigma^2_{r_p} = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j\neq i}^{N} \omega_i \omega_j \sigma_{ij}
\]

Naïve Diversification

- Invest in $N$ assets
- Invest $1/N$ in each asset
  - equally weighted portfolio

\[
\sigma^2_p = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N} \sum_{i=1}^{N} \sum_{j\neq i}^{N} \sigma_{ij}
\]

\[
= \frac{1}{N} \text{AVERAGE}(\sigma^2) + \frac{N-1}{N} \text{AVERAGE}(\sigma_{ij})
\]
Naïve Diversification

One risky and riskless asset

- Invest $\omega$ in risky and $(1- \omega)$ in riskless
- Expected Return of Portfolio?

$$E[r_p] = \omega E[r] + (1-\omega)r_f$$

$$= r_f + \omega(E[r]-r_f)$$

- Portfolio Standard Deviation

$$\sigma_p = \sqrt{\omega^2 \sigma^2}$$

$$= \omega \sigma$$
Example

- Risk free rate: 7%
- Risky asset
  - expected return: 15%
  - standard deviation: 22%
- Premium on risky asset: 8%
- Invest half in each
  - expected return: 11%, standard dev: 11%

Varying Investment Proportions
Punchline

• Linear relationship between expected return and standard deviation
• Capital Allocation Line (CAL)
• slope: extra reward for risk

\[ \sigma_p = \omega \sigma \rightarrow \omega = \frac{\sigma_p}{\sigma} \]

\[ E[\bar{r}_p] = r_f + \omega(E[\bar{r}] - r_f) \]

\[ E[\bar{r}_p] = r_f + \frac{(E[\bar{r}] - r_f)}{\sigma} \sigma_p \]

Which risky asset to choose?

• combine risk free with some index
  – easy to do, low transactions costs
  – CML
• If you could choose one risky, want highest slope, or...

\[ \text{Max}_{\bar{r}_{\text{port}}} \left( \frac{(E[\bar{r}_{\text{port}}] - r_f)}{\sigma_{\text{port}}} \right) \]
Two risky assets

- Invest $\omega$ in asset 1 and $(1 - \omega)$ asset 2

\[
E[r_p] = \omega E[r_1] + (1 - \omega)E[r_2]
\]

\[
\sigma_p^2 = \omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2 + 2\omega(1 - \omega)\sigma_{12}
\]

\[
= \omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2 + 2\omega(1 - \omega)\rho_{12}\sigma_1\sigma_2
\]
So far...

- Combining riskless and one risky
  - straight line
  - choosing risky: maximize slope
- Multiple risky assets
  - two assets, correlation coefficient is key
  - with lots of assets, average covariance matters

Multiple Risky Assets

- Vector notation
  - \( \omega \): N by 1 vector of asset weights
  - \( \Sigma \): variance-covariance matrix of asset returns, N by N matrix
  - \( \mu \): N by 1 vector of expected returns
  - \( \mathbf{1} \): N by 1 vector of ones
Portfolios of Risky Assets

• Portfolio Expected Return:

\[ E[\tilde{r}_p] = \omega^\prime \mu \]

• Portfolio Variance:

\[ \sigma_p^2 = \omega^\prime \Sigma \omega \]

• Constraint

\[ 1 = \omega^\prime 1 \]
Frontier of Risky Assets

- Minimize portfolio variance subject to
  - mean constraint
  - changing constraint traces out entire curve
- portfolio constraint
  \[ \text{Min} \omega \sigma_m^2 = \omega \Sigma \omega \]
  \[ \text{st.} \]
  \[ \omega^\prime \mu = m \]
  \[ \omega^\prime t = 1 \]

Properties of Solution

- Can be done in Excel using solver

- First Order Conditions:
  \[ \Sigma^{-1} \omega^* = 0.5 \lambda_1 \mu + 0.5 \lambda_2 t \]
  \[ \omega^* \mu = m \]
  \[ \omega^* \prime t = 1 \]
Two Fund Separation

• Useful to compute results
• All solutions to the problem are a combination of two portfolios
  – minimum variance portfolio
  – arbitrary optimal portfolio

Minimum Variance Portfolio

• Lowest possible portfolio variance
• Ignore mean constraint
• Use solver in Excel

\[
\begin{align*}
\text{Min } & \sigma_{mv}^2 = \omega' \Sigma \omega \\
\text{s.t. } & \\
\omega' \iota &= 1
\end{align*}
\]
How to generate frontier

- Solve for global minimum variance portfolio: \( r_{mv} \)
  - compute the mean, \( m_{mv} \)
- Pick arbitrary mean, and solve for optimal portfolio at that mean: \( r_m \)
- Any other mean, \( m_n \)
  - weight \( w \), st. \( w m_{mv} + (1-w)m = m_n \)

Example

- 3 assets
- global minimum variance portfolio
  - \([.2,.5,.3]\)
  - mean at this point 4%
- efficient portfolio with mean of 10%
  - \([.5,.1,.4]\)
Want optimal portfolio w/ mean of 7%?

Summary

• Mean Variance Analysis
• why/why not
• two asset portfolios
• role of covariance
• two fund result
Next Time

• Add investment in riskless asset…
• Excel example?
• Practical issues
• Equilibrium: The CAPM and APT
• References
  – Chapters 9, 10 in text
  – Kritzman on Factor models.