Mean Variance Analysis

Implementation Issues and
Equilibrium

Objectives

• Enough information to implement in Excel
• Practical Issues
• Why factor models are useful for this
• Combining many risky assets with riskless
• CAPM
Multiple Risky Assets

- Vector notation
- $\omega$: N by 1 vector of asset weights
- $\Sigma$: variance-covariance matrix of asset returns, N by N matrix
- $\mu$: N by 1 vector of expected returns
- $\iota$: N by 1 vector of ones

Portfolios of Risky Assets

- Portfolio Expected Return:
  \[ E[\tilde{r}_p] = \omega^\prime \mu \]
- Portfolio Variance:
  \[ \sigma^2_p = \omega^\prime \Sigma \omega \]
- Constraint
  \[ 1 = \omega^\prime \iota \]
Feasible Portfolios

Frontier of Risky Assets

- Minimize portfolio variance subject to
  - mean constraint
    - changing constraint traces out entire curve
  - portfolio constraint

\[
\begin{align*}
\min_{\omega} \sigma_m^2 &= \omega' \Sigma \omega \\
\text{st.} \quad &
\omega' \mu = m \\
&
\omega' 1 = 1
\end{align*}
\]
Properties of Solution

- Can be done in Excel using solver

- First Order Conditions:
  \[ \Sigma \omega^* = 0.5 \lambda_1 \mu + 0.5 \lambda_2 t \]
  \[ \omega^* \cdot \mu = m \]
  \[ \omega^* \cdot t = 1 \]

Two Fund Separation

- Useful to compute results
- All solutions to the problem are a combination of two portfolios
  - minimum variance portfolio
  - arbitrary optimal portfolio
Minimum Variance Portfolio

- Lowest possible portfolio variance
- Ignore mean constraint
- Use solver in Excel

\[
\text{Min}_{\omega} \sigma_{mv}^2 = \omega'\Sigma\omega
\]
\[s.t.
\omega'1 = 1\]

How to generate frontier

- Solve for global minimum variance portfolio: \(r_{mv}\)
  - compute the mean, \(m_{mv}\)
- Pick arbitrary mean, and solve for optimal portfolio at that mean: \(r_m\)
- Any other mean, \(m_n\)
  - weight \(w\), st. \(w m_{mv} + (1-w)m = m_n\)
Example

- 3 assets
- global minimum variance portfolio
  - [.2,.5,.3]
  - mean at this point 4%
- efficient portfolio with mean of 10%
  - [.5,.1,.4]

Want optimal portfolio w/ mean of 7%?
Variance Calculations

Useful result

- Covariance global minimum variance portfolio and any other portfolio = variance of global minimum variance
Additional Constraints

- Often, fund can’t
  - short sell
  - over-invest in sectors
  - use all assets (Green Funds)
- Solution: constraints on problem
- Two-fund result goes away
- Curve moves inside unconstrained frontier
Practical Issues

- You need to get inputs
  - mean vector
  - variance-covariance matrix
- Historical Data?
  - Sampling errors can cause problems

Sampling Error

- Sample average to estimate means and variance covariance matrix

\[
\hat{m} = \frac{1}{T} \sum_{t=1}^{T} \tilde{r}_t
\]
\[
\hat{s}^2 = \frac{1}{T} \sum_{t=1}^{T} (\tilde{r}_t - \hat{m})^2
\]

- Estimates have noise!
Example

- Monthly iid returns, stocks
  - 1% per month on average return
  - standard deviation: 4.33 per month
- 2 years of data to estimate returns
  - standard deviation of estimate

\[
\sigma(\hat{m}) = \frac{\sigma_m}{\sqrt{T}} \\
= \frac{4.33}{\sqrt{24}} \approx 0.884
\]

So What

- 1 standard deviation confidence band
  - 1% ± 0.884%
- Optimal portfolio weights are very sensitive to mean estimates…
- Standard deviation is much more precisely estimated with same amount of data
Solutions

- Don’t only rely on statistics
  - fundamental analysis
  - average with priors
  - use equilibrium restrictions (Black and Litterman)

Estimating covariance matrix

- Lots of parameters in covariance matrix
  - with 10 assets, 45 parameters!
- Don’t want too long data series
  - stationary issues
- Need to invert covariance matrix
- Solution
  - Factor models….
Factor Models and Portfolio Optimization

• Why
  – reduce number of parameters in var-cov matrix to estimate
• Plausible model

Single Factor Model

• For each asset,
  \[ \tilde{r}_{i,t} = a + \beta_i \tilde{F}_t + \tilde{\epsilon}_{i,t} \]

• Asset Variance
  \[ \sigma_i^2 = \beta_i^2 \sigma_F^2 + \sigma_{\tilde{\epsilon}_i}^2 \]
Covariances of Assets

- Calculate the covariances among assets
  \[ \text{Cov}(\tilde{r}_i, \tilde{r}_j) = \text{Cov}(\beta_i \tilde{F} + \tilde{\varepsilon}_i, \beta_j \tilde{F} + \tilde{\varepsilon}_j) = \beta_i \beta_j \sigma^2_{\tilde{F}} \]

- Putting it all together...

Ways to get factors

- Factor Analysis
- Economic Factors
  - market portfolio
  - interest rates or term structure
  - inflation
  - oil prices, etc.
Combining with Riskless

• Find efficient portfolio with largest slope

\[
\frac{\left( \hat{E}[\hat{r}_{\text{port}}] - r_f \right)}{\sigma_{\text{port}}}
\]
So far...

- Given mean and covariances can find optimal risky position
- Adding riskless
- Practical issues
  - sensitivity
  - where to get inputs

Equilibrium

- Why worry about equilibrium?
  - Benchmark
  - Pricing results
- If we agree on picture
  - all pick same risky position
  - market portfolio!
So What

- easy to figure out optimal strategy
- Pricing result
- For all investments, efficient or not calculate beta relative to market

\[ E[\tilde{r}_i] = r_f + \beta_{i,\text{market}} \left( E[\tilde{r}_{\text{market}}] - r_f \right) \]
Discussion Questions

• You can’t get higher return than market if you don’t hold an efficient portfolio
• Stocks are riskier than bonds: all stocks should have higher expected return than risk free bonds
• You can’t time the market

More Questions...
Yet More

- IBM has a beta of 1
- Is it as risky as market?

Summary

- how to implement
- implementation issues
- factor models again
- equilibrium: CAPM
Next Time

• APT…

• Applications of models: performance evaluation

• Readings:
  – Chapters 11, 24, text
  – Sharpe on Sharpe ratio (readings packet)