Objectives

- Nonparallel yield curve shifts
  - factor models
  - hedging applications
- Adding Time
- Options
  - what?

Recent TS (Bloomberg)
Main Points

• Non-parallel shifts
• Independence of yield changes across bonds?
• Modify ‘hedging’ tools to deal with this

Basic Idea

• Make yields ‘related’ statistically
• Work with continuously compounded yields

\[
P_{\text{bond}} = \frac{C}{\exp(r)} + \frac{C}{\exp(2 \times r)} + \cdots \frac{C + F}{\exp(\tau \times r)} \\
= \frac{C}{\exp(y_{\text{bond}})} + \frac{C}{\exp(2 \times y_{\text{bond}})} + \cdots \frac{C + F}{\exp(\tau \times y_{\text{bond}})}
\]
Example

- Reference bond: 5 year zero
- 5 year yield = 10%

\[ P_{5\text{year}} = \frac{1}{\exp(0.1 \times 5)} = 0.607 \]

Duration with continuous compounding

- Works just like before!
- (Slight) adjustment for continuous compounding

\[ D = \sum \frac{cf_i}{\exp(y \times i) \times P} \times i \]
Convexity Formula

- Convexity of zero = maturity squared
- example: 5 year PDB, convexity=
- In general, weighted average

\[ Conv = \sum \left( \frac{c_f}{\exp(y) \times P} \right) \times \frac{1}{2} \times \frac{c_f}{\exp(y \times t) \times P} \times t^2 + \cdots \]

One year yield changes

- Change in the 5 year bond yield
  - proportional to change in 1 year bond yield
  - proportionality factor is 1.1
  - 1% change in 1 year

\[ \frac{\Delta P_{\text{5 year}}}{P_{\text{5 year}}} = -S(\Delta r) \]
\[ = -5 \times (1.1 \times \Delta r) \]
\[ = -5 \times 1.1 \times 1\% \]
\[ = -5.5\% \]
How would you hedge now?

- Change in 5 year = 5 times 1.1 times change in 1 year
- ‘Adjusted’ duration = 5.5
- x% in 5 year and (1-x)% in 1 year

\[ \frac{\Delta P_{\text{port}}}{P_{\text{port}}} = x \frac{\Delta P_{\text{5 year}}}{P_{\text{5 year}}} + (1-x) \frac{\Delta P_{\text{1 year}}}{P_{\text{1 year}}} \]

\[ = x \times -D_s \times 1.1 + (1-x) \times (-1) \]

\[ = x \times (-5.5) + (1-x) \times (-1) \]

Example

- Have a liability with a duration of 3
- Hedge with the one year and 5 year PDB
- Using regular duration?
- Using adjusted duration of 5 year = 5.5

\[ D_p = xD_5 + (1-x)D_1 \]

\[ 3 = x5.5 + (1-x)1 \]

\[ \rightarrow x = \frac{2}{4.5} \]
Litterman and Scheinkman

- Extend this idea to multiple sources of movement of yields
- Statistical fitting of sensitivities
- Applied in practice!

What to do in Data?

- Look for ‘commonalties’ in movements of various zero yields
- Estimate via ‘factor’ model or regression
- Interpretation

\[ r_{i,t} = A_i + \beta_{1,i} F_{1,t} + \cdots + \beta_{K,i} F_{K,i,t} + \epsilon_{i,t} \]
Terminology

• $F_{k,t}$: factors
  – K of them
  – affect yields systematically
  – zero mean, variance 1, 0 correlation with each other

• $\beta_{k,i}$: factor sensitivity or loading
  – how much does shock to factor k affect yield i

Terminology (cont.)

• $\varepsilon_{i,t}$: residuals, unexplained movements
  – diversifiable

• $A_i$: Yield with no factor shocks

• Factor Analysis
Estimation Method

Effects of Factor Shocks

change in yield

maturity
Estimates of Model

<table>
<thead>
<tr>
<th>Maturity</th>
<th>R-squared</th>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Factor 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.995</td>
<td>79.5</td>
<td>17.2</td>
<td>3.3</td>
</tr>
<tr>
<td>2 years</td>
<td>0.982</td>
<td>93.4</td>
<td>2.4</td>
<td>4.2</td>
</tr>
<tr>
<td>5 years</td>
<td>0.988</td>
<td>98.2</td>
<td>1.1</td>
<td>0.7</td>
</tr>
<tr>
<td>14 years</td>
<td>0.984</td>
<td>86.2</td>
<td>11.5</td>
<td>2.2</td>
</tr>
</tbody>
</table>

Example

- One factor, F
- P=$1 million, D=4 years, PDB
- Use 8 year PDB
- Factor loading: 4 years = 1.1, 8 years = 1.2
- Pick weight so that factor adjusted duration = 0
Solution

\[ 0 = x \times 4 \times 1.1 + (1 - x) \times 8 \times 1.2 \]

\[ \rightarrow x = \frac{9.6}{5.2}, 1 - x = \frac{-4.4}{5.2} \]

- Go long 229% in 4 year bond and short 129% in 8 year bond
- What about reverse?

Factor Adjusted Convexity

- What about ‘big’ factor changes
  - adjust convexity
- Adjusted convexity of 4 year PDB with sensitivity of 1.1?

\[ C_{\text{adjusted}} = C(\beta_1^2) \]

\[ = 4^2 \times 1.1^2 \]

\[ = 19.36 \]
Portfolio

- Invest x in bond 1 and (1-x) in bond 2
- Portfolio Adjusted Convexity?

\[ C_{port} = xC_1\beta_1^2 + (1 - x)C_2\beta_2^2 \]

Including Time Value or Putting it all together

- Up to now, ignored time value
- Change in position over small unit of time
- Taylor-series

\[
\frac{\Delta P}{P} = \frac{y\Delta t}{P} - D_p\beta_p\Delta F + 0.5C_p\beta_p^2(\Delta F)^2 \\
= A_p\Delta t - D_p\beta_p\Delta F + 0.5C_p\beta_p^2(\Delta F)^2
\]
Portfolio Movements

- Invest $x$ in 1 year PDB and $(1-x)$ in 2 year PDB
- 50% in each of the bonds
  - with sensitivities 1.1, and 1.2
- intercepts of 0.001 and 0.002
Computations

\[
\frac{\Delta P}{P} = (xA_1 + (1 - x)A_2) - (xD_1\beta_1 + (1 - x)D_2\beta_2)\Delta F \\
+ 0.5(xC_1\beta_1^2 + (1 - x)C_2\beta_2^2)(\Delta F)^2
\]

\[
\approx (0.5(0.001) + 0.5(0.002))(0.5(1.1) + 0.5(2)(1.2))\Delta F \\
+ 0.5(0.5(1.1)^2 + 0.5(2)^2(1.2)^2)(\Delta F)^2
\]
Adjusted Convexity vs. Yield

- spread position: short bullet and long barbell
- Modified duration = 0
- Positive convexity
- Intercept?

Introduction to options

- Options are everywhere
  - traded options: basic and exotic
  - interest rate options
  - many securities have embedded options
    - callable debt, convertible debt, warrants, mortgages, etc.
  - Many real investments have option features
Terminology

- Call
- Put
- European style
- American style

- Exercising:
- Strike (exercise) price:
- expiration date
- $C_t$: price of call, $P_t$: put price, $S_t$: stock price
- $r$: risk free rate
Sequence of events for American call

- Now: purchase call for $C_0$
- Exercise time, $T$
  - $S_T > X$: exercise and pay $X$ for stock
  - $S_T < X$: don’t exercise

$$C_T = \begin{cases} S_T - X, & \text{if } S_T > X \\ 0, & \text{otherwise} \end{cases}$$

$$= \max(0, S_T - X)$$

Puts

- Similar timing
- Buying put not the same as writing call

$$P_T = \begin{cases} 0, & \text{if } S_T > X \\ X - S_T, & \text{if } S_T \leq X \end{cases}$$

$$= \max(0, X - S_T)$$
Payoff Diagrams

- Long stock
- Short stock

European call

- Payoffs
- Stock price
- Long call
- X
Summary

- Factor models for bonds
  - what
  - why
  - how to apply
- Adding time
- Introduction to options

Next Time

- Strategies involving options
- Basic option valuation principles
  - Chapter 21, text