Objectives

- Valuing longer maturity options
  - how
  - hedging
  - implications of strategy
- Dynamic trading strategies
  - buy and hold, constant proportion
  - CPPI

Valuing multi-period options

- Same basic idea: replicate final payoffs
- Example
  - $S_0=10$
  - $u=1.5, d=0.5$
  - $r=1.1$ per period
- Value call with 2 periods, $X=5$
Stock payoffs

10
  \( \begin{array}{c} \text{u} \\ \downarrow \\ \text{d} \end{array} \) 15 \( \begin{array}{c} \text{u} \\ \downarrow \\ \text{d} \end{array} \)

\( \begin{array}{c} \text{d} \\ \downarrow \\ \text{u} \end{array} \)

\( \begin{array}{c} \text{d} \\ \downarrow \\ \text{u} \end{array} \)

22.5

7.5

2.5

Call option payoffs

\( \begin{array}{c} \text{C}_0 \\ \downarrow \\ \text{d} \end{array} \)

\( \begin{array}{c} \text{u} \\ \downarrow \\ \text{C}_u \end{array} \)

\( \begin{array}{c} \text{d} \\ \downarrow \\ \text{C}_d \end{array} \)

\( \begin{array}{c} \text{u} \\ \downarrow \\ \text{d} \end{array} \)

\( \begin{array}{c} \text{u} \\ \downarrow \\ \text{d} \end{array} \)

Max(22.5-5,0) = 17.5

Max(7.5-0,0) = 2.5

Max(2.5-5,0) = 0
Valuation

• Work backwards through tree.
• Replicate option value at each node
• Results
  – strategy of stock and bond with same payoffs as option
  – value of strategy = value of option

After initial ‘u’, or $C_u$

• # shares: $\alpha_u$, # bonds: $\beta_u$ bonds
• $C_{uu} = 17.5$, $C_{ud} = 2.5$, $S_{uu} = 22.5$, $S_{ud} = 7.5$, $S_u = 15$

\[
\begin{align*}
  uu & : \quad \alpha_u 22.5 + \beta_u 1.1 = 17.5 \\
  ud & : \quad \alpha_u 7.5 + \beta_u 1.1 = 2.5 \\
  \rightarrow \alpha_u & = 1, \beta_u = -5 / 1.1 \\
  Cost & = 15 - 5 / 1.1 = 10.45 = C_u
\end{align*}
\]
After initial ‘d’

- # shares: \( a_d \), # bonds: \( b_d \) bonds
- \( C_{du} = 2.5, C_{uu} = 0, S_{du} = 7.5, S_{dd} = 2.5, S_d = 5 \)

\[
du: \quad \alpha_d 7.5 + \beta_d 1.1 = 2.5 \\
dd: \quad \alpha_d 2.5 + \beta_d 1.1 = 0 \\
\rightarrow \alpha_d = 0.5, \beta_d = -1.25 / 1.1 \\
Cost = 0.5(5) - 1.25 / 1.1 = 1.36 = C_d
\]

Initial value

- Replicate the value of the option in both the ‘u’ and ‘d’ states next period
- \( C_u = 10.45, C_d = 1.36, S_u = 15, S_d = 5, S_0 = 10 \)

\[
u: \quad \alpha 15 + \beta 1.1 = 10.45 \\
d: \quad \alpha 5 + \beta 1.1 = 1.36 \\
\rightarrow \alpha = 0.909, \beta = -3.2 / 1.1 \\
Cost = 0.909(10) - 3.2 / 1.1 = 6.2 = C_0
\]
Important points

- Probabilities of ‘u’ and ‘d’ don’t matter
- 2 securities, but 3 final outcomes for option
  - how can it work?
  - Need #securities greater than or equal to # states
- Dynamic trading

The positions

- Initial holdings
  - stock: 0.909, bonds -3.2/1.1
- Price rise
  - stock 1, bond -5/1.1
- Price fall
  - stock 0.5, bond -1.25/1.1
- trend chasing?
Key Points

- Replication over time
- Probabilities don’t matter
- Dynamic trading can make many complicated payoffs
- Valuation technique
- Hedging technique

Black Scholes

- Same basic model:
- Let time between periods get ‘small’
- Parameters
  - risk-free rate
  - stock volatility (‘u’ and ‘d’)
  - dividends
  - maturity of option
Dynamic trading strategies

• Sharpe and Perold

• Objectives:
  – understand what kind of dynamic strategies you could follow and why?
  – Mechanics of re-balancing
Exposure Diagram 60/40 Buy-Hold

Constant Mix Strategies

- Invest constant % in risky asset
- Risk tolerance increases in wealth
- Dynamic Strategy
  - you need to trade to maintain the position
Example

- Start with $100
- $60 in stocks and $40 in Bills
- want to maintain constant weight

<table>
<thead>
<tr>
<th>Case</th>
<th>Stock Value</th>
<th>Stock Value</th>
<th>Bill Value</th>
<th>Asset Value</th>
<th>% in Stocks</th>
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</thead>
<tbody>
<tr>
<td>Initial</td>
<td>$100</td>
<td>$60</td>
<td>$40</td>
<td>$100</td>
<td>60</td>
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<tr>
<td>Market Drop</td>
<td>$90</td>
<td>$54</td>
<td>$40</td>
<td>$94</td>
<td>57.4</td>
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<tr>
<td>After rebalancing</td>
<td>$90</td>
<td>$56.4</td>
<td>$37.6</td>
<td>$94</td>
<td>60</td>
</tr>
</tbody>
</table>

Main Points

- Must rebalance to maintain constant weight
- Buy stocks when price falls and vice-versa when stock rises
- Issue of when to trade
  - typical rule: wait for % of price move
  - depends on volatility
Comparison: buy and hold and constant proportion

<table>
<thead>
<tr>
<th>Case</th>
<th>Stock Value</th>
<th>Bill Value</th>
<th>Asset Value</th>
<th>% Stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>100</td>
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<td>40</td>
<td>100</td>
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<tr>
<td>Price drop</td>
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<td>54</td>
<td>40</td>
<td>94</td>
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<td>100.27</td>
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Main Points

- Flat Volatile markets
  - constant proportion tends to do better

- Trends
  - better with `buy and hold’

- Trading costs?
Constant Proportion Strategies

- Dollars in stock = m (assets - floor)
- m: multiplier
- m > 1: constant proportion portfolio insurance
- Who?

Example

- $100 wealth
- $75 floor
- multiplier of 2
- Initial cushion: 100-75=25
- Investment in stocks = 2(25)=50
- Preferences of investor?
Exposure Diagram

Example of Strategy

<table>
<thead>
<tr>
<th>Case</th>
<th>Mkt Stock</th>
<th>Bill</th>
<th>Asset</th>
<th>Cush</th>
<th>% in Stock</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>100</td>
<td>50</td>
<td>50</td>
<td>100</td>
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<tr>
<td>Rebalance</td>
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<tr>
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<td>53.24</td>
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<tr>
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<td>120</td>
<td>60.5</td>
<td>45.5</td>
<td>105.5</td>
<td>30.5</td>
</tr>
</tbody>
</table>
Main Points

• Sell stock in bear
• Buy stock in bull
• Relative to buy and hold
  – gain on downside
  – cost on upside
  – high enough, dominates buy and hold

What assets?

• Stocks and bonds only?
Underlying Assumptions

• When won’t this work?

Payoff Shape

• Straight line: buy and hold

• Concave: buy stocks as they fall
  – constant mix

• Convex: buy stocks as they rise
  – CPPI and constant mix
Summary

• Dynamic replication of options
  – hedging
  – pricing
• Dynamic strategies
  – what kinds and why

Next Time

• Mean-variance analysis
  – basic assumptions and why?
• Introduction to implementation
• References
  – Text: Chapters 7-9 (brief review)
  – Kritzman on optimization (readings)