7. ORDINARY DIFFERENTIAL EQUATION

7.1 Introduction

DE ⊂ ODE  # of indep. var = 1
PDE ⊂  # of indep. var ≥ 2

(dependent variable
independent variable

Linear ODE ← we will deal with this type only in this class.

Non linear ODE

Def. of Linear ODE

\[ a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = f(x) \]

(linear combination of
\( y, y', y'', \ldots, y^{(n)} \))
Question: Which ones are linear ODE?

1. \( \frac{dy}{dx} = x + 1 \)
2. \( \frac{dy}{dx} = y + 1 \)
3. \( \frac{dy}{dx} = x^3 + 1 \)
4. \( \frac{dy}{dx} = (x^3 + 1)y \)
5. \( \frac{dy}{dx} = y^2 + 1 \)
6. \( \frac{dy}{dx} = \sin y \)

Answer: (1), (2), (3), (4)

e.g.) Mass-spring-damper system

\[ m \ddot{x} + c \dot{x} + kx = F(t) \quad \text{linear} \]

e.g.) Swinging Pendulum

\[ \frac{d^2 \theta}{dt^2} = -\frac{g}{l} \sin \theta \]

Nonlinear, but if \( \theta \approx 0 \)

\[ \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \]

Linearized version.

\[ \frac{d^2 \theta}{dt^2} = -\frac{g}{l} \theta \]
Higher order linear ODE

\[ \Rightarrow \quad \text{Coupled 1st order ODEs} \]

This conversion is necessary in order to apply Euler’s or Runge-Kutta methods to solving a higher order ODE.

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = F(x) \]

\[ \Rightarrow \quad n^\text{th order ODE} \]

\[ y_0 = y, \]
\[ y_1 = y', \]
\[ y_2 = y'', \]
\[ y_3 = y''' \]
\[ \vdots \]
\[ y_{n-1} = y^{(n-1)} \]

\[ \Rightarrow \quad \text{Coupled 1st order ODEs} \]

\[ \begin{cases} 
  y_0' = y_1 \\
  y_1' = y_2 \\
  y_2' = y_3 \\
  \vdots \\
  y_{n-1}' = f(t) - a_0 y_0 - a_1 y_1 - \cdots - a_{n-1} y_{n-2} \end{cases} \]

Vector form \[ y' = f(x, y) \]

\[
\begin{array}{c}
\text{indep. var.} \\
\uparrow \\
\text{dep. var.} 
\end{array}
\]
2nd order ODE $\rightarrow$ 1st order ODE

Original 2nd order ODE:

$$m x'' + c x' + k x = F(t)$$

State variables: $y_0, y_1$

$$\begin{cases}
y_0' = y_1 \\
y_1' = \frac{F(t) - ky_0 - cy_1}{m}
\end{cases}$$

This function is also a vector

$y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$, $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \frac{F(t) - ky_0 - cy_1}{m} \end{bmatrix}$
**ODE Integration Scheme Summary**

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Most popular RK method for practical applications.
7.3 Euler's Method

Input

\[ y' = f(x, y) \]

vectors \( \rightarrow \) indep \( \rightarrow \) dep var.

The 1st order derivative (or slope) is a function of both \( x \) and \( y \).

Euler's method (1st order Taylor approx)

\[ y_{i+1} = y_i + f(x_i, y_i)h + O(h^2) \]

\[ \text{local truncation error} \]

in one step, \( x_i \to x_{i+1} \)

\[ \text{global truncation error} \quad \text{(always greater than local)} \]

\[ \# \text{ of steps} \times O(h^2) \]

(local)

\[ = O\left(\frac{1}{h}\right) \times O(h^2) \]

\[ = O(h) \]

\( O(h^2) \) is better than \( O(h) \)
Heun's Method
(predictor-corrector approach)

\[ y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^\circ)}{2} \]

Heun's method:

\[ y_{i+1}^\circ = y_i + f(x_i, y_i)h \]

\[ y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^\circ)}{2} h \]

corrector

predictor
Midpoint Method

\[ f(x_i, y_i) \]

Midpoint: \( x_i \), \( x_i + \frac{h}{2} \), \( x_{i+1} \)

\[
\begin{align*}
    y_{i+\frac{1}{2}} &= y_i + f(x_i, y_i) \cdot \frac{h}{2} \\
    y_{i+1} &= y_i + f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) \cdot h \\
    x_{i+\frac{1}{2}} &= x_i + \frac{h}{2}
\end{align*}
\]

Both Heun's and midpoint methods are examples of the 2nd order Runge-Kutta method.
7.3 Runge-Kutta Methods

- Higher order linear ODE
  \[ y' = f(x, y) \]

  - Independent variable
  - Dependent variable

\[ \frac{dy}{dx} = x^2 + y \quad \text{(linear)} \]
\[ \frac{dy}{dx} = y^2 + x \quad \text{(non-linear)} \]

- Euler
- Heun
- Mid-Point

Examples of R-K methods (one step methods)
- Weighted average
- Sub-steps

To estimate a representative slope more accurately
Generalized form of RK solutions:

\[ y_{i+1} = y_i + \phi(x_i, y_i, h) h. \]

\[ \phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n \]

- RK-1st

\[ k_1 = f(x_i, y_i) \]

- RK-2nd

\[ k_2 = f(x_i + 0.5h, y_i + 0.5h k_1) \]

- RK-3rd

\[ k_3 = f(x_i + 0.5h, y_i + 0.5h + k_2) \]

- RK-4th

\[ k_4 = f(x_i + h, y_i + h k_1 + \frac{h^2}{2} k_2 + \frac{h^3}{3} k_3) \]

\[ \phi = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]

\[ \text{Note:} \]

- k's are recurrence relationships.

- That is, k_1 appears in the \( f \) for k_2, which appears in the \( f \) for k_3, and so forth.
RK-1st

$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$

where $k_1 = f(x_i, y_i)$
$k_2 = f(x_i + p_i h, y_i + g_{11} k_1 h)$

End

RK-2nd

$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$

where $k_1 = f(x_i, y_i)$
$k_2 = f(x_i + p_i h, y_i + g_{11} k_1 h)$

4 unknowns: $a_1, a_2, p_1, g_{11}$

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Read textbook p.497
Box 25.1

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\[\begin{align*}
    a_1 + a_2 &= 1 \\
    a_2 p_1 &= \frac{1}{2} \\
    a_2 g_{11} &= \frac{1}{2}.
\end{align*}\]

\[\begin{align*}
    a_1 &= 1 - a_2 \\
    p_1 &= \frac{1}{2} a_2 \\
    g_{11} &= \frac{1}{2} a_2
\end{align*}\]

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Third Edition

$\times$ typo in the textbook

$\times$ equivalent to the 2nd order Taylor series approx.

Also (25.32) on p.696 is a typo.

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This is why the 2nd order RK is exact to quadratic func.

The common basic strategy underlying all the Runge-Kutta methods
Three simultaneous eqs for four unknowns 
(one more unknown than the # of eqs) 
→ no unique set of solutions. 
→ by assuming a value for one 
we can determine the other three 

\[ a_2 = \frac{1}{2} \text{ (Heun's)} \]
\[ a_1 = \frac{1}{2}, \quad p_1 = \frac{3}{4} \quad n = 1 \]
\[ y_{i+1} = y_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \]

where 
\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + h, y_i + k_1 h) \]

\[ a_2 = 1 \text{ (Mid point)} \]
\[ a_1 = 0, \quad p_1 = \frac{3}{4} \quad n = 1 \]
\[ y_{i+1} = y_i + k_2 h \]

where 
\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}) \]

\[ a_2 = \frac{2}{3} \text{ (Ralston) ← minimum trunc. error} \]
\[ a_1 = \frac{1}{3}, \quad p_1 = \frac{3}{4} \quad n = 3/4 \]
\[ y_{i+1} = y_i + \left( \frac{1}{3} k_1 + \frac{2}{3} k_2 \right) h \]

where 
\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + \frac{3}{4} h, y_i + k_1 \frac{3}{4} h) \]
\[ f(x, y) \]

ex.) \( y' = 2x \quad \rightarrow \quad y = x^2 \).

\[ y' = 6. \]

\[ y' = 4. \quad \text{exact solution to the slope} \]

\[ y' = 3 \]

\[ \phi = 3 \]

\[ \chi_i = 0 \quad \chi_{i+1} = 3 \quad h = 3. \]

Heun's

\[ k_1 = f(0, 0) = 0 \]

\[ k_2 = f(3, 0) = 6 \]

\[ \phi = \frac{1}{2} k_1 + \frac{1}{2} k_2 = 3 \]

Mid point

\[ k_1 = f(0, 0) = 0 \]

\[ k_2 = f\left(\frac{3}{2}, 0\right) = 3 \]

\[ \phi = k_2 = 3 \]

Ralston

\[ k_1 = f(0, 0) = 0 \]

\[ k_2 = f\left(\frac{9}{4}, 0\right) = \frac{9}{2} \]

\[ \phi = \frac{1}{3} k_1 + \frac{2}{3} k_2 = \frac{3}{3} \cdot \frac{9}{2} = 3 \]
**RK-1st**

\[ k_1 \rightarrow \phi \rightarrow k_2 \]

\[ x_i, x_{i+1} \]

**RK-2nd**

- **Heun**
  \[ a_1 = a_2 = \frac{1}{2} \]

- **Mid-point**
  \[ a_1 = 0, a_2 = 1 \]

- **Ralston**
  \[ a_1 = \frac{1}{3}, a_2 = \frac{2}{3} \]
\[ y_{i+1} = y_i + \frac{k_1 + 2k_1 + 2k_3 + k_4}{6} h \]

where \[ k_1 = f(x_i, y_i) \]
\[ k_2 = f(x_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}) \]
\[ k_3 = f(x_i + \frac{h}{2}, y_i + k_2 \frac{h}{2}) \]
\[ k_4 = f(x_i + h, y_i + k_3 h) \]

\[ a_1 = \frac{1}{6}, a_2 = \frac{1}{3}, a_3 = \frac{1}{3}, a_4 = \frac{1}{6} \]

This formula is equivalent to the 4th order Taylor series approximation.

Local error (per step) \( O(h^5) \)

Global error (per \( \frac{1}{h} \) steps) \( O(h^4) \)

Exact to the quintic function.