Exam #4

NAME: __________________________________________

Please circle your recitation section.

A  A. Winger  TuTh 12:30  PH A18C
B  A. Winger  TuTh 1:30  DH 1217
C  D. Berol  TuTh 3:30  DH 1211

- This exam consists of 6 problems. It is your responsibility to make sure you have all the pages.

- No notes or books may be consulted during the exam.

- No calculators may be used during the exam.

- All solutions that you submit must be your own work. You may not look at or copy the work of others during this exam.

- Show your work. No credit will be given for unsupported incorrect answers.

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1. (14 points) Evaluate each of the following:

(a) \[ \int_{0}^{1} (3e^{-3x} + \sqrt{x}) \, dx. \]

Answer:
\[
\int_{0}^{1} \left(3e^{-3x} + x^{1/2}\right) \, dx = \left[ -e^{-3x} + \frac{2}{3}x^{3/2} \right]_{0}^{1} = \left( -e^{-3} + \frac{2}{3} \right) - (-e^0)
\]
\[= \frac{-1}{e^3} + \frac{5}{3}\]

(b) \[ \int_{-1}^{2} \left(2x + \sqrt{7} - 3x^2\right) \, dx. \]

Answer:
\[
\int_{-1}^{2} \left(2x + \sqrt{7} - 3x^2\right) \, dx = \left[ x^2 + \sqrt{7}x - x^3 \right]_{-1}^{2} = \left( 4 + 2\sqrt{7} - 8 \right) - \left( 1 - \sqrt{7} + 1 \right)
\]
\[= 3\sqrt{7} - 6\]
2. (24 points) Evaluate each of the following:

(a) \[ \int x e^{(x^2 + 5)} \, dx \]
Answer: Let \( u = x^2 + 5 \), so \( du = 2x \, dx \).
\[
\int x e^{(x^2 + 5)} \, dx = \frac{1}{2} \int e^{(x^2 + 5)}(2x) \, dx = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C
\]
\[
= \frac{1}{2} e^{(x^2 + 5)} + C
\]

(b) \[ \int_0^1 x \sqrt{4x^2 + 5} \, dx \]
Answer: Let \( u = 4x^2 + 5 \), so \( du = 8x \, dx \).
\[
\int_0^1 x \sqrt{4x^2 + 5} \, dx = \frac{1}{8} \int_0^1 (4x^2 + 5)^{\frac{1}{2}} (8x) \, dx = \frac{1}{8} \int_{x=0}^{x=1} u^{\frac{1}{2}} \, du
\]
\[
= \left( \frac{1}{12} u^{\frac{3}{2}} \right)_{x=0}^{x=1} = \left( \frac{1}{12} (4x^2 + 5)^{\frac{3}{2}} \right)_{x=0}^{x=1} = \left( \frac{27}{12} \right) - \left( \frac{5\sqrt{5}}{12} \right)
\]
\[
= \frac{27 - 5\sqrt{5}}{12}
\]

(c) \[ \int \frac{3e^x}{e^x + 4} \, dx \]
Answer: Let \( u = e^x + 4 \), so \( du = e^x \, dx \).
\[
\int \frac{3e^x}{e^x + 4} \, dx = 3 \int \frac{1}{u} \, du = 3 \ln |u| + C = 3 \ln |e^x + 4| + C
\]
\[
= 3 \ln (e^x + 4) + C
\]
3. (18 points) Evaluate each of the following:

(a) \( \int \frac{x^3}{1 + x^2} \, dx \)

Answer: Let \( u = 1 + x^2 \), so \( du = 2x \, dx \) and \( x^2 = u - 1 \).

\[
\int x^3 (1 + x^2)^{-1} \, dx = \frac{1}{2} \int x^2 (1 + x^2)^{-1} (2x) \, dx = \frac{1}{2} \int (u - 1)u^{-1} \, du \\
= \frac{1}{2} \int (1 - u^{-1}) \, du = \frac{1}{2} (u - \ln |u|) + C \\
= \frac{1}{2} (1 + x^2) - \frac{1}{2} \ln (1 + x^2) + C
\]

(b) \( \frac{d}{dx} \left[ \int_{e^x}^{x} \frac{1}{1 + t^6} \, dt \right] \)

Answer: Using the Fundamental Theorem of Calculus and the chain rule, we have

\[
\frac{d}{dx} \left[ \int_{e^x}^{x} \frac{1}{1 + t^6} \, dt \right] = \frac{d}{dx} \left[ \int_{0}^{x} \frac{1}{1 + t^6} \, dt - \int_{0}^{e^x} \frac{1}{1 + t^6} \, dt \right] \\
= \frac{1}{1 + (x)^6} \frac{d}{dx} [x] - \frac{1}{1 + (e^x)^6} \frac{d}{dx} [e^x] \\
= \frac{1}{1 + x^6} - \frac{e^x}{1 + e^6x}
\]
4. (14 points) The demand equation for a product is 

\[ p = 9 - q^2, \]

and the supply equation is 

\[ p = 1 + q^2. \]

Determine the consumers' surplus and producers' surplus at market equilibrium.

Answer:
First, we find the point of market equilibrium.

\[ 9 - q^2 = 1 + q^2 \Rightarrow q^2 = 4 \Rightarrow q = \pm 2. \]

Since \( q \) must be non-negative, we find that the point at market equilibrium is \((2, 5)\).

\[
\text{consumers' surplus} = \int_0^2 (9 - q^2 - 5) \, dq = \left( 4q - \frac{1}{3}q^3 \right) \bigg|_0^2 = \left( 8 - \frac{8}{3} \right) - 0 = \frac{16}{3} \approx 5.33
\]

\[
\text{producers' surplus} = \int_0^2 (5 - 1 - q^2) \, dq = \left( 4q - \frac{1}{3}q^3 \right) \bigg|_0^2 = \left( 8 - \frac{8}{3} \right) - 0 = \frac{16}{3} \approx 5.33
\]
5. (14 points) Find the area of the region bounded by the curves $x = y^2$, $y = 2 - x$ and $y = 0$.

As drawn on the board:

Method 1 (vertical slices):

$$A = \int_{0}^{1} [x^2 - 0] \, dx + \int_{1}^{2} [2 - x - 0] \, dx = \left( \frac{2}{3} x^3 \right)_{0}^{1} + \left( \frac{2}{2} x - \frac{1}{2} x^2 \right)_{1}^{2}$$

$$= \frac{2}{3} + 4 - 2 - 2 + \frac{1}{2} = \frac{7}{6}$$

Method 2 (horizontal slices):

$$A = \int_{0}^{1} [(2 - y) - y^2] \, dy = \left( 2y - \frac{1}{2} y^2 - \frac{1}{3} y^3 \right)_{0}^{1}$$

$$= 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$$
6. (16 points) Find the volume of the solid generated when the region bounded by the curves $y = 2 - x^2$ and $y = 1$ is revolved about the $x$-axis.

Method 1 (vertical slices/washer method): The volume of each thin washer is

$$\left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right] dx = \left[\pi \left(2 - x^2\right)^2 - \pi(1)^2\right] dx$$

$$= \pi \left(x^4 - 4x^2 + 3\right) dx$$

Total volume is

$$V = \int_{-1}^{1} \pi \left(x^4 - 4x^2 + 3\right) dx = \pi \left(\frac{1}{5}x^5 - \frac{4}{3}x^3 + 3x\right) \bigg|_{-1}^{1}$$

$$= \frac{\pi}{5} - \frac{4\pi}{3} + 3\pi + \frac{\pi}{5} - \frac{4\pi}{3} + 3 = \frac{56\pi}{15}$$

Method 2 (horizontal slices/cylindrical shells): The volume of each cylindrical shell is

$$2\pi rh dy = 2\pi(y) \left[\sqrt{2-y} - \left(-\sqrt{2-y}\right)\right] dy = 4\pi y\sqrt{2-y} dy$$

Total volume is (let $u = 2 - y$, so $du = -dy$ and $y = 2 - u$)

$$V = \int_{1}^{2} 4\pi y\sqrt{2-y} dy = -4\pi \int_{y=1}^{y=2} (2-u)u^2 du = \left(\frac{-16\pi}{3}u^3 + \frac{8\pi}{5}u^2\right)\bigg|_{y=1}^{y=2}$$

$$= \left(\frac{-16\pi}{3}(2-y)^2 + \frac{8\pi}{5}(2-y)^2\right)\bigg|_{y=1}^{y=2} = \frac{16\pi}{3} - \frac{8\pi}{5} = \frac{56\pi}{15}$$