Handout #3:  
An Application of Integration to Economics 

Suppose that the equation \( p = f(q) \) provides the price \( p \) at which consumers will purchase (demand) \( q \) units. Further suppose that the equation \( p = g(q) \) provides the price \( p \) at which the manufacturers will sell (supply) \( q \) units. The graph of \( p = f(q) \) is called a demand curve, and the graph of \( p = g(q) \) is called a supply curve. The point \((q_0, p_0)\) where these two curves intersect (i.e. \( p_0 = f(q_0) = g(q_0) \)) is called the point of equilibrium. When the price for a product is \( p_0 \) per unit, the number of units demanded by the consumer is exactly equal to the number of units supplied by the manufacturer. Basically, the price \( p_0 \) is the price at which demand and supply are in perfect balance; in such a situation, we say that the market is at equilibrium.

Assume that the market is at equilibrium when the price per unit is \( p_0 \). When looking at a demand curve, one typically sees that there are consumers who would be willing to pay more than \( p_0 \) per unit. These consumers are receiving a benefit from the lower price \( p_0 \); they are spending less money per unit than they are actually willing to spend. The total gain to the consumers willing to pay more than \( p_0 \) can be represented by the quantity

$$\text{consumers' surplus} = \int_0^{q_0} [f(q) - p_0] \, dq,$$

where \( q_0 \) is such that \( p_0 = f(q_0) = g(q_0) \). Geometrically, the consumers’ surplus is the area below the demand curve and above the horizontal line \( p = p_0 \).

When looking at a supply curve, one notices that there are manufacturers that are willing to supply the product at a price less than \( p_0 \). These producers are benefiting from the higher equilibrium price \( p_0 \). The total gain for these producers can be represented by the quantity

$$\text{producers' surplus} = \int_0^{q_0} [p_0 - g(q)] \, dq.$$

Geometrically, the producers’ surplus is the area above the supply curve and below the horizontal line \( p = p_0 \).
Let’s look at an example. Suppose that the demand function for a product is given by
\[ p = f(q) = 100 - 0.05q, \]
where \( p \) is the price per unit (in dollars) for \( q \) units. Suppose also that the supply function is
\[ p = g(q) = 10 + 0.1q. \]
We want to find the consumers’ surplus and the producers’ surplus at market equilibrium.

The first thing to do is determine the point of equilibrium \( (q_0, p_0) \). This can be found by solving simultaneously the system of equations \( p = 100 - 0.05q \) and \( p = 10 + 0.1q \):

\[
100 - 0.05q = 10 + 0.1q \\
0.15q = 90 \\
q = 600.
\]

So \( q_0 = 600 \) and
\[ p_0 = f(q_0) = g(q_0) = 70. \]

The point of equilibrium is \( (70, 600) \).

Now, we use the formulas for the consumers’ and producers’ surplus. The consumers’ surplus is
\[
\int_0^{q_0} [(f(q) - p_0)] dq = \int_0^{600} [(100 - 0.05q) - (70)] dq
= \left[ 30q - \frac{0.05}{2}q^2 \right]_0^{600} = 9000.
\]

The producers’ surplus is
\[
\int_0^{q_0} [p_0 - g(q)] dq = \int_0^{600} [(70) - (10 + 0.1q)] dq
= \left[ 60q - \frac{0.01}{2}q^2 \right]_0^{600} = 18,000.
\]

So, the consumers’ surplus is $9000 and the consumers’ surplus is $18,000.