For problems 1 & 2, calculate eigenvectors and eigenvalues by hand. For all other problems, feel free to use MathCAD or your calculator.

Coupled Systems of First Order Differential Equations

1. (15 points) For the following set of coupled, first order differential equations with the initial condition that \( y_1(0) = 1 \) and \( y_2(0) = 0 \):

\[
\begin{align*}
\dot{y}_1 &= a y_1 + b y_2 \\
\dot{y}_2 &= -2 y_1 + y_2
\end{align*}
\]

a. If \( a = 0 \) and \( b = -1 \), find the general and particular solution for \( y(t) \). Plot \( y_1(t) \) versus \( y_2(t) \) and comment on the form of the solution space.

b. If \( a = 5 \) and \( b = 2 \), find the general and particular solution for \( y(t) \). Plot \( y_1(t) \) versus \( y_2(t) \) and comment on the form of the solution space.

c. If \( a = -1 \) and \( b = 1 \), find the general and particular solution for \( y(t) \). Plot \( y_1(t) \) versus \( y_2(t) \) and comment on the form of the solution space.

2. (15 points) For the following homogeneous second order differential equation:

\[
y'' + 2 y' + y = 0; \ y(0) = 1, \ y'(0) = 0
\]

a. Solve this as a constant coefficient, second order equation and find the general and particular solutions for \( y(t) \).

b. Using the substitutions \( y_1(t) = y(t) \) and \( y_2(t) = y_1(t) \), rewrite the second order equation as a system of two coupled first order equations (see pg. 156 Kreyszig).

c. Solve the system in part (b) for the general and particular solutions of \( y \). Write the particular solutions for \( y_1(t) \) and \( y_2(t) \).

d. Are the solutions from parts (a) and (b) the same?

3. (15 points) Given the following linear third order differential equation with initial conditions:

\[
y^{(3)} + 2 y'' + 9 y' + 18 y = 0; \ y(0) = 1, \ y'(0) = 0, \ y''(0) = 0
\]

where \( y^{(3)} \) denotes the third derivative of the unknown function \( y(x) \).

a. Convert this single third order system into a system of three coupled first order differential equations.
b. Solve for the general solution for the three unknown functions \( y_1(t) \), \( y_2(t) \) and \( y_3(t) \).

c. Find the particular solution for each function and plot \( y_1(t) \), \( y_2(t) \) and \( y_3(t) \).

d. Plot the particular solution for the unknown function \( y(t) \).

**Modeling of Coupled First Order Systems**

4. **(15 points)** A mixing system is set up to dilute the product of a reaction to different levels for sale to a variety of customers. Two mixing vessels (tanks) are attached and a mixture of solvent plus product (call it A) flows through the system at a constant volumetric flow rate, \( F \). The mixture in vessel 1 flows into vessel 2 and then back into vessel 1. The second tank is initially filled with solvent \( (C_2(0) = 0) \) while the first tank is initially filled with a concentrated solution \( C_1(0) = 10 \text{ gm A/m}^3 \), and the two vessels have different volumes.

![Diagram of mixing system](image)

a. Develop a system of coupled equations that describes the rate of change of concentration of A in each of the vessels.

b. For the case that \( V_1 = 50 \text{ m}^3 \), \( V_2 = 200 \text{ m}^3 \) and \( F = 10 \text{ m}^3/\text{min} \), solve the system of coupled equations for the concentration of A as a function of time.

c. Plot the concentration of A in each tank as a function of time. How long does it take for the system to reach steady state? If you want to sell product at a concentration of 3 gm A/m\(^3\), at what time do you stop the mixing, cap mixer 2 and sell the product?

5. **(15 points)** You are designing a batch reactor (see picture). In the well-mixed reactor, A reacts to form the intermediate B at a rate proportional to the concentration of A in the reactor. Simultaneously, B reacts to form the product C at a rate proportional to the concentration of B.

![Reaction diagram](image)

- Rate of A consumed in the reaction \( A \rightarrow B \) is \( k_1 C_A(t) \) where \( k_1 = 0.5 \text{ min}^{-1} \).
- Rate of B consumed in the reaction \( B \rightarrow C \) is \( k_2 C_B(t) \) where \( k_2 = 0.3 \text{ min}^{-1} \).

a. Perform separate balances on each component and develop models for the rate of change of concentration of each component \( (C_A, C_B, \text{ and } C_C) \).
b. Solve the coupled system of equations for the concentrations as a function of time. Write the general solution for the unknown concentration vector \( \mathbf{c} = \begin{bmatrix} C_A & C_B & C_C \end{bmatrix}^T \).

c. If the reactor is initially filled with solvent and A (i.e., \( C_A(t=0) = 10 \) mol/L and \( C_B(t=0) = C_C(t=0) = 0 \) mol/L), plot the concentrations of A, B and C in the reactor as a function of time. How long does the system take to reach steady state?

6. **(15 points)** Three tanks are attached in series as shown below. All three tanks have the same constant cross sectional area.

![Diagram showing three tanks in series](image)

\[
\begin{align*}
F_1(t) &= \frac{h_1(t) - h_2(t)}{R_1} \\
F_2(t) &= \frac{h_2(t) - h_3(t)}{R_2} \\
F_3(t) &= \frac{h_3(t)}{R_3}
\end{align*}
\]

- \( R_1 = 2 \) hr/m²
- \( R_2 = 1 \) hr/m²
- \( R_3 = 1 \) hr/m²

a. Perform mass balances on each of the three tanks and develop a model for the rate of change of height of liquid in each tank.

b. Write these three mass balances as a matrix equation \( \mathbf{h}' = \mathbf{A} \mathbf{h} \) and identify all terms in the vector \( \mathbf{h} \) and matrix \( \mathbf{A} \).

c. Solve the system of coupled equations for \( \mathbf{h}(t) \).

d. If the first and third tanks are initially full (\( h = 10 \) m) and the middle tank is initially empty, plot the heights of each tank as a function of time.

e. At what time does the middle tank reach its maximum height?