Homework 1

1. Take your favorite system for deduction in propositional logic, and write $\varphi \vdash_p \psi$ for "p is a deduction of ψ from an assumption of φ". Define a *category of proofs*, with objects the formulas $\varphi, \psi, \ldots$ and arrows the deductions,

   $$p : \varphi \to \psi \quad \text{iff} \quad \varphi \vdash_p \psi$$

   Composition is by the obvious substitution of a conclusion for an assumption in a deduction, and identity arrows are trivial proofs.

   Show that this *is* a category. Show that this is *not* a preorder category.

2. Verify that, for a fixed set A, the operations $X \mapsto X^A$ and $X \mapsto A^X$ are, respectively, co- and contravariant functors on the category of sets.

3. Let $\mathbf{C}$ be small.

   (a) Show that Hom is a functor $\mathrm{Hom} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Cat}$.

   (b) Show that the slice category construction determines a functor $\mathbf{C}/(-) : \mathbf{C} \to \mathbf{Cat}$ by composition.

4. For the algebraically inclined:

   (a) For any set X, define the *free monoid on* X by:

   $$F(X) = \sum_{n \in \mathbb{N}} X^n \qquad \text{(disjoint sum)}$$

   So, informally, elements of $F(X)$ are finite tuples $(x_1, \ldots, x_m)$ of elements of X. The monoid unit is the empty tuple $u = (-)$, and the multiplication is concatination of tuples,

   $$(x_1, \ldots, x_m) \cdot (y_1, \ldots, y_n) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$$

   Prove that this *is* a monoid.

   (b) Show that $F(X)$ is *free on* X in the following sense: there is a function $\eta : X \to F(X)$, namely $\eta(x) = (x)$, such that for any monoid M and any function $f : X \to M$, there is a unique monoid homomorphism $\overline{f} : F(X) \to M$ such that $\overline{f} \circ \eta = f$.

   (c) Show that there is a functor $F : \mathbf{Sets} \to \mathbf{Mon}$ taking a set X to the free monoid $F(X)$. (Hint: you can do it by hand, but the most elegant proof uses part b).