Computability in ergodic theory

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Ergodic theory

A *discrete dynamical system* consists of a structure, $\mathcal{X}$, and an map T from $\mathcal{X}$ to $\mathcal{X}$:

- Think of the underlying set of $\mathcal{X}$ as the set of states of a system.
- If x is a state, Tx gives the state after one unit of time.

In ergodic theory, $\mathcal{X}$ is assumed to be a finite measure space $(X, \mathcal{B}, \mu)$:

- $\mathcal{B}$ is a σ-algebra (the "measurable subsets").
- μ is a σ-additive measure, with $\mu(X) = 1$.

T is assumed to be a *measure-preserving transformation*, i.e. $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$.

Ergodic theory

Call $(X, \mathcal{B}, \mu, T)$ a *measure-preserving system*.

- These can model physical systems (e.g. Hamilton's equations preserve Lebesgue measure).
- They can model probabilistic processes.
- They have applications to number theory and combinatorics.

The metamathematics of ergodic theory

Ergodic theory emerged from seventeenth century dynamics and nineteenth century statistical mechanics.

Since Poincaré, the emphasis has been on characterizing structural properties of dynamical systems, especially with respect to long term behavior (stability, recurrence).

Today, the field uses structural, infinitary, and nonconstructive methods that are characteristic of modern mathematics.

These are often at odds with computational concerns.

The metamathematics of ergodic theory

Central questions:

- To what extent can the methods and objects of ergodic theory be given a direct computational interpretation?
- How can we locate the "constructive content" of the nonconstructive methods?
- Can we extract additional qualitative and quantitative information from nonconstructive proofs?

I will focus on two case studies:

- the von Neumann and Birkhoff ergodic theorems
- the Furstenberg-Zimmer structure theorem, and Furstenberg's ergodic-theoretic proof of Szemerédi's theorem

The ergodic theorems

Consider the orbit x, Tx, $T^2x, \ldots$, and let $f : \mathcal{X} \to \mathbb{R}$ be some measurement. Consider the averages

$$\frac{1}{n}(f(x) + f(Tx) + \ldots + f(T^{n-1}x)).$$

For each $n \geq 1$, define $A_n f$ to be the function $\frac{1}{n}\sum_{i<n} f \circ T^i$.

Theorem (Birkhoff). For every f in $L^1(\mathcal{X})$, $(A_n f)$ converges pointwise almost everywhere, and in the L^1 norm.

A space is *ergodic* if for every A, $T^{-1}(A) = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$.

If $\mathcal{X}$ is *ergodic*, then $(A_n f)$ converges to the constant function $\int f \; d\mu$.

The ergodic theorems

Recall that $L^2(\mathcal{X})$ is the Hilbert space of square-integrable functions on $\mathcal{X}$ modulo a.e. equivalence, with inner product

$$(f, g) = \int fg \; d\mu$$

Theorem (von Neumann). For every f in $L^2(\mathcal{X})$, $(A_n f)$ converges in the L^2 norm.

A measure-preserving transformation T gives rise to an isometry T on $L^2(\mathcal{X})$,

$$Tf = f \circ T.$$

Riesz showed that the von Neumann ergodic theorem holds, more generally, for any nonexpansive operator T on a Hilbert space (i.e. satisfying $\|Tf\| \leq \|f\|$ for every f in $\mathcal{H}$.)

Bounding the rate of convergence

Can we compute a bound on the rate of convergence of $(A_n f)$ from the inital data (T and f)?

In other words: can we compute a function $r : \mathbb{Q} \to \mathbb{N}$ such that for every rational $\varepsilon > 0$,

$$\|A_m f - A_{r(\varepsilon)} f\| < \varepsilon$$

whenever $m \geq r(\varepsilon)$?

Krengel (et al.): convergence can be arbitrarily slow. But computability is a different question.

Note that the question depends on suitable notions of computability in analysis.

Observations

If $(a_n)_{n \in \mathbb{N}}$ is a sequence of reals that decreases to 0, no matter how slowly, one can compute a bound on the rate of convergence from (a_n).

But there are bounded, computable, decreasing sequences (b_n) of rationals that do not have a computable limit.

There are also computable sequences (c_n) of rationals that converge to 0, with no computable bound on the rate of convergence.

Conclusion: at issue is not the *rate* of convergence, but its *predictability*.

A negative result

Theorem (A-Simic). There are a computable measure-preserving transformation of $[0, 1]$ under Lebesgue measure and a computable characteristic function $f = \chi_A$, such that if $f^* = \lim_n A_n f$, then $\|f^*\|_2$ is not a computable real number.

In particular, f^* is not a computable element of $L^2(\mathcal{X})$, and there is no computable bound on the rate of convergence of $(A_n f)$ in either the L^2 or L^1 norm.

A positive result

Theorem (A-Gerhardy-Towsner). Let T be a nonexpansive operator on a separable Hilbert space and let f be an element of that space. Let $f^* = \lim_n A_n f$. Then f^*, and a bound on the rate of convergence of $(A_n f)$ in the Hilbert space norm, can be computed from f, T, and $\|f^*\|$.

In particular, if T arises from an ergodic transformation T, then f^* is computable from T and f.

A constructive mean ergodic theorem

It turns out that we can say more, even in situations where there is no computable bound on the rate of convergence.

The assertion that the sequence $(A_n f)$ converges can be represented as follows:

$$\forall \varepsilon > 0 \, \exists n \, \forall m \geq n \, (\|A_m f - A_n f\| < \varepsilon).$$

This is classically equivalent to the assertion that for any function K,

$$\forall \varepsilon > 0 \, \exists n \, \forall m \in [n, K(n)] \, (\|A_m f - A_n f\| < \varepsilon).$$

A constructive mean ergodic theorem

Theorem (A-G-T). Let T be any nonexpansive operator on a Hilbert space, let f be any element of that space, and let $\varepsilon > 0$, and let K be any function. Then there is an $n \geq 1$ such that for every m in $[n, K(n)]$, $\|A_m f - A_n f\| < \varepsilon$.

In fact, we provide a bound on n expressed solely in terms of K and $\rho = \|f\|/\varepsilon$. Notably, the bound is independent of X and T.

As special cases, we have the following:

- If $K = n^{O(1)}$, then $n(f, \varepsilon) = 2^{2^{O(\rho^2 \log\log\rho)}}$.
- If $K = 2^{O(n)}$, then $n(f, \varepsilon) = 2^{1}_{O(\rho^2)}$.
- If $K = O(n)$ and T is an isometry, then $n(f, \varepsilon) = 2^{O(\rho^2 \log \rho)}$.

A constructive pointwise ergodic theorem

The following is classically equivalent to the pointwise ergodic theorem:

Theorem (A-G-T). For every f in $L^2(\mathcal{X})$, $\lambda_1 > 0$, $\lambda_2 > 0$, and K there is an $n \geq 1$ satisfying

$$\mu(\{x \mid \max_{n \leq m \leq K(n)} |A_n f(x) - A_m f(x)| > \lambda_1\}) \leq \lambda_2.$$

We provide explicit bounds on n in terms of f, λ_1, λ_2, and K.

Bishop's *upcrossing inequalities* provides another constructive interpretation of the pointwise ergodic theorem.

Proof mining

The Riesz proof of the mean ergodic theorem shows that if $\mathcal{H}$ is a Hilbert space and T is nonexpansive, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ where

- $\mathcal{M} = \{f \mid Tf = f\}$
- $\mathcal{N} = \overline{\{Tg - g \mid g \in \mathcal{H}\}}$

If f is in $\mathcal{M}$, $A_n f = f$ for every n.

If f is in $\mathcal{N}$, then $A_n f \to 0$.

Thus $A_n f$ converges to $P_{\mathcal{M}}(f)$, the projection of f onto $\mathcal{M}$.

It is the use of a projection that makes the proof nonconstructive.

Proof mining

Our constructive proof was obtained using "proof mining" methods, developed chiefly by Ulrich Kohlenbach and his students.

- General metamathematical results, based on Gödel's Dialectica translation, guarantee that such information can always be extracted from proofs in certain formal systems.
- Our constructive version of the mean ergodic theorem is an instance of Kreisel's "no-counterexample interpretation."
- The method of eliminating the use of a projection comes from Kohlenbach, "Elimination of Skolem functions for monotone formulas."
- The uniformity we obtain is predicted by Gerhardy and Kohlenbach, "Generalized metatheorems on the extractability of uniform bounds in functional analysis."

Proof mining

Independently, Terence Tao hit upon the no-counterexample method of obtaining constructive / quantitative versions of nonconstructive statements, providing for a passage from "soft" analysis to "hard" analysis. He referred to such phenomena as "metastability."

The idea played a central role in his "Norm convergence of multiple ergodic averages for commuting transformations."

Similar quantitative methods were central to his work with Ben Green on arithmetic progressions in the primes.

Our proofs can are a form of "energy incrementation" argument. These relationship between the infinitary and quantitative methods needs to be better understood.

Ergodic Ramsey theory

Let us consider an applications of ergodic theory to combinatorics.

Theorem (van der Waerden). If one colors the natural numbers with finitely many colors, then there are arbitrarily long monochromatic arithmetic progressions.

The theorem has a finitary (Π_2) statement:

Theorem. For every k and r there is an n large enough such that if one colors elements of the set $\{1, \ldots, n\}$ with r colors, there is a monochromatic arithmetic progression of length k.

van der Waerden proved this in 1927. Furstenberg and Weiss presented an elegant proof using topological dynamics in 1978.

Szemerédi's theorem

Szemerédi's theorem is a "density" version of van der Waerden's theorem.

Szemerédi's Theorem. Every set S of natural numbers with positive upper Banach density has arbitrarily long arithmetic progressions.

Equivalently:

Theorem. For every k and $\delta > 0$, there is an n large enough, such that if S is any subset of $\{1, \ldots, n\}$ with density at least δ, then S has an arithmetic progression of length k.

History

- 1936: Conjectured by Erdös and Turán
- 1952: Roth proved it for $k = 3$.
- 1969: Szemerédi proved it for $k = 4$.
- 1974: Szemerédi proved the full theorem.
- 1977: Furstenberg
  - gave an equivalent ergodic-theoretic statement,
  - provided a structural analysis of ergodic measure-preserving systems, and
  - used the latter to give a proof.
- 1979: Furstenberg and Katznelson used the structure theorem to give a streamlined proof of an even stronger result.
- 2001: Gowers gave a new proof of Szemerédi's theorem, with elementary bounds.
- 2004: Tao and Green used quantitative ergodic-theoretic methods to prove that there are arbitrarily long arithmetic progressions in the primes.

Logical analysis

The fact that powerful infinitary methods can yield explicit combinatorial results deserves logical analysis.

Recall the central questions:

- To what extent can the methods and objects of ergodic theory be given a direct computational interpretation?
- How can we locate the "constructive content" of the nonconstructive methods?
- Can we extract additional qualitative and quantitative information from nonconstructive proofs?

Furstenberg correspondence

Suppose there were a sequence of subsets S_m of $\{0, \ldots, m-1\}$ of density $\delta > 0$, with no arithmetic progression of length k.

Consider the spaces $X_m = \{0, \ldots, 2m-1\}$ with uniform distribution and shift map $Tx = x + 1 \bmod 2m$. Then for every m and $n < m$,

$$S_m \cap T^{-n} S_m \cap T^{-2n} S_m \cap \ldots \cap T^{-(k-1)n} S_m = \emptyset.$$

A compactness argument yields a space $(X, \mathcal{B}, \mu)$ and set S that gives a counterexample to the following:

Theorem. For any measure-preserving system $(X, \mathcal{B}, \mu, T)$, any set S of positive measure, and any k, there is an n such that

$$\mu(S \cap T^{-n} S \cap T^{-2n} S \cap \ldots \cap T^{-(k-1)n} S) > 0.$$

In fact, this theorem is equivalent to Szemerédi's theorem.

Two distinct behaviors

A measure-preserving system is *weak mixing* if we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0.$$

A weak mixing system exhibits a high degree of randomness.

A measure-preserving system is *compact* if it has the property that for every measurable set A, the orbit

$$\{A, T^{-1}, T^{-2}A, T^{-3}A, \ldots\}$$

is totally bounded, i.e. has compact closure.

A compact system exhibits a high degree or regularity.

Lack of randomness implies order

Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be a measure-preserving system.
(Henceforth, assume T is invertible.)

Lemma (Koopman-von Neumann). If $\mathcal{X}$ is not weak mixing, it has a nontrivial compact T-invariant factor.

Three ways of thinking of a T-invariant factor:

- $(X, \mathcal{B}', \mu, T)$, for a T-invariant sub-σ-algebra $\mathcal{B}' \subseteq \mathcal{B}$
- a homomorphic image, or quotient, of $(X, \mathcal{B}, \mu, T)$
- T-invariant subspace of $L^2(\mathcal{X})$, containing the constant functions and closed under max.

The Furstenberg-Zimmer structure theorem

The notions of compactness and weak mixing relativize to factors.

Lemma (Furstenberg, Zimmer). If a system $(X, \mathcal{B}, \mu, T)$ is not weak mixing relative to a factor $\mathcal{B}'$, there there is an intermediate factor $\mathcal{B}''$ such that $(X, \mathcal{B}'', \mu, T)$ is compact relative to $(X, \mathcal{B}', \mu, T)$.

We can iterate this, taking unions at limit stages. If the system is separable, the process comes to an end at a countable ordinal.

The Furstenberg-Zimmer structure theorem

Theorem (Furstenberg, Zimmer). Let $(X, \mathcal{B}, \mu, T)$ be any measure-preserving system. Then there is a transfinite increasing sequence of factors $(\mathcal{B}_\alpha)_{\alpha \leq \gamma}$ such that:

- $\mathcal{B}_0$ is the trivial factor.
- For each $\alpha < \gamma$, $(X, \mathcal{B}_{\alpha+1}, \mu, T)$ is compact relative to (X, B_α, μ, T).
- For each limit $\lambda \leq \gamma$, $\mathcal{B}_\lambda = \cup_{\alpha<\lambda} \mathcal{B}_\alpha$.
- Either $\mathcal{B}_\gamma = \mathcal{B}$, or $(X, \mathcal{B}, \mu, T)$ is weakly mixing relative to $(X, \mathcal{B}_\gamma, \mu, T)$.

Each $\mathcal{B}_\alpha$ is said to be *distal*, and $\mathcal{B}_\gamma$ is said to be the *maximal distal factor*.

The Furstenberg-Katznelson-Ornstein proof

Say a set A is SZ if for every k, $\mu(A) > 0$ implies

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i<n} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-(k-1)n}A) > 0.$$

Say a factor is SZ if every element is SZ.

This is a strengthening of the desired conclusion for S.

The property of being SZ:

- holds the trivial factor;
- is maintained under compact extensions;
- is maintained under limits; and
- is maintained under weak mixing extensions.

Diagnosing the nonconstructivity

What makes the proof nonconstructive?

The correspondence theorem may seem suspect. But the recursion-theoretic complexity is mild, and proof-theoretic techniques are well-equipped to handle such uses of compactness.

The real culprit is the iterative construction of factors: each step requires taking ergodic limits and projections.

Diagnosing the nonconstructivity

Theorem (Beleznay and Foreman). For any countable ordinal α, there is a separable measure-preserving system such that any Furstenberg-Zimmer tower has height at least α.

The theorem effectivizes: if X codes a measure-preserving system, the height of the tower is less than or equal to $\omega_1^{CK,X}$. The αth level is computable in $H^X_{2\cdot\alpha}$.

We suspect that this is sharp, at least for limit α. This means that the Furstenberg-Zimmer tower is a wildly noncomputable object.

Townser and I have shown, however, that "sufficiently" weak mixing factors occur lower down, before level ω^{ω^ω}.

In fact, Furstenberg's original argument requires only k levels.

Diagnosing the nonconstructivity

To summarize:

- The Furstenberg-Katznelson-Ornstein proof requires a long transfinite tower (axiomatic strength around ID_1).
- A weakening by Avigad and Towsner requires a tower of height at most ω^{ω^ω} (predicative theories suffice).
- Furstenberg's original proof requires a tower of height k (axiomatic strength around PA).
- A quantitative version due to Tao has a similar structure, but with explicit bounds (axiomatic strength around PRA).
- Gowers' elementary bounds presumably go through in elementary function arithmetic.

Gowers' work requires new ideas; there is no way they can be "mined" from the ergodic-theoretic methods. But the connection between Tao's proof and the Furstenberg methods should be better understood.

Diagnosing the nonconstructivity

Recall that the measure space coming out of the Furstenberg construction can be viewed as a "limit" of finite spaces. Tao's quantitative proof simply uses a sufficiently large finite space.

One difficulty: constructions in the limit do not correspond to constructions in the finite spaces. For example, a factor in the limit is not a limit of factors.

Tao considers complexity-bounded approximations to the "true" ergodic-theoretic factors, for example, finite factors where the number of atoms is bounded independent of n.

It would be helpful to have a cleaner connection to the infinitary argument.

Conclusions

Goals:

- A better understanding of the relationship between the infinitary ("soft") and finitary, quantitative ("hard") methods.
- Infinitary methods that are better suited to finitary problems.
- Additional information from proofs using the infinitary methods.
- An understanding as to how and where logical strength can be avoided, and where it is necessary.

There is a lot to do:

- Dynamical systems represents represent an uneasy tension between structural and computational concerns.
- Applications to combinatorics, in particular, require both structural ideas and quantitative information.

Some references

Associated papers and talks can be found on my web page:

- "Fundamental notions of analysis in subsystems of second-order arithmetic" (with Ksenija Simic)
- "Local stability of ergodic averages" (with Philipp Gerhardy and Henry Towsner)
- "Functional interpretation and inductive definitions" (with Henry Towsner)
- "The metamathematics of ergodic theory"
- "Metastability in the Furstenberg-Zimmer tower" (with Henry Towsner)