Lean Together 2021: Teaching with Proof Assistants

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Teaching with Lean

Theorem Proving in Lean (with Soonho Kong and Leo de Moura)

- An introduction to dependent type theory and core Lean syntax and semantics.
- Advantages: it explains the theory and the system from the bottom up.
- Disadvantages: it does not include mathlib theories and tactics, it doesn't get around to substantial formalization.

Mathematics in Lean (with Kevin Buzzard, Robert Lewis, and Patrick Massot)

- Focuses on mathlib and formalizing interesting mathematics as quickly as possible.
- It's in progress (60+ pages already).

Teaching with Lean

Logic and Proof (with Robert Lewis and Floris van Doorn)

- An introduction to symbolic logic, informal mathematical proof, and formal proof.
- Intended for freshmen and sophomores in mathematics, computer science, and other fields (even philosophy or business).
- It doesn't assume anything beyond high-school mathematics.

Logic and Proof

Goals of the course:

- Teach students to write ordinary mathematical proofs.
- Teach students how to use symbolic logic (to make assertions, prove assertions, and specify properties).
- Teach students to use Lean, in service to the other two goals.

The three strands are largely independent.

The point is that the three topics hang together, and there is some synergy.

Logic and Proof

Question. Let $\equiv$ be an equivalence relation on a set A. For every element a in A, let $[a]$ be the set of elements $\{c \in A \mid c \equiv a\}$, that is, the set of elements of A that are equivalent to a. Show that for every a and b, $[a] = [b]$ if and only if $a \equiv b$.

Student answer. Assume $[a] = [b]$. This means that the set of elements $c \in A$ such that $c \equiv a$ is equal to the set of elements $c \in A$ such that $c \equiv b$. Thus everything that is equivalent to a is also equivalent to b. But by transitivity of $\equiv$, $a \equiv c$ and $c \equiv b$ implies that $a \equiv b$.

Moreover, assume $a \equiv b$. Then any $c \in A$ that is $\equiv$ to a will also be $\equiv$ to b by transitivity, and any $c \in A$ that is $\equiv$ to b will also be $\equiv$ to a by transitivity. So $[a] = [b]$.

Logic and Proof

Question. Let f be any function from X to Y, and let g be any function from Y to Z. Show that if $g \circ f$ is injective, then f is injective.

Give an example of functions f and g as above, such that $g \circ f$ is injective, but g is not injective.

Student answer (to first part). Assume that $g \circ f$ is injective. Then by definition, for all $a, b \in X$, we have that $(g \circ f)(a) = (g \circ f)(b) \implies a = b$.

Now assume that there exist some $x, y \in X$ such that $f(x) = f(y)$. Then we have $(g \circ f)(x) = (g \circ f)(y)$, which implies $x = y$ by the injectivity of $g \circ f$. So f is injective by definition.

Logic and Proof

The hope is that doing things formally helps students become more structured and precise with their language.

```
def disj (A B : set U) : Prop :=
∀ x, x ∈ A → x ∈ B → false

example (A B C D : set U)
    (h1 : disj A B) (h2 : C ⊆ A) (h3 : D ⊆ B) :
  disj C D :=
assume x,
assume e1 : x ∈ C,
assume e2 : x ∈ D,
have r1 : x ∈ A, from h2 e1,
have r2 : x ∈ B, from h3 e2,
show false, from h1 x r1 r2
```

Logic and Proof

```
variables A B C : Type
variables (f : A → B) (g : B → C)

example (h : injective (g ∘ f)) : injective f :=
assume x1: A,
assume x2: A,
assume h1: f x1 = f x2,
have h2: g (f x1) = g (f x2), by rw h1,
show x1 = x2, from h h2

example (h : surjective (g ∘ f)) : surjective g :=
assume z : C,
exists.elim (h z) $
assume x : A,
assume h1: (g (f x)) = z,
exists.intro (f x) h1
```

Logic and Proof

Observations:

- Make it clear that there are three distinct languages:
  - ordinary mathematics
  - symbolic logic
  - formal proof languages

  Students will not get them confused.
- The parallel developments seemed to help. Students could “see” an exists elimination or an or elimination in an informal proof.
- Students liked the course. There was no clear favorite among the topics: some liked Lean more than the other parts, some less.