Microsoft Research
Summit 2022

The Digital Revolution in Mathematics

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New technology for mathematics

Lean as a theorem prover: a platform for

- defining mathematical objects,
- stating theorems,
- and writing complex proofs.

Lean as a programming language:

- a performant functional programming language, with
- means for writing specifications and proving that programs satisfy them.

Combining the two brings

- computational methods to mathematical reasoning, and
- mathematical reasoning to computation.

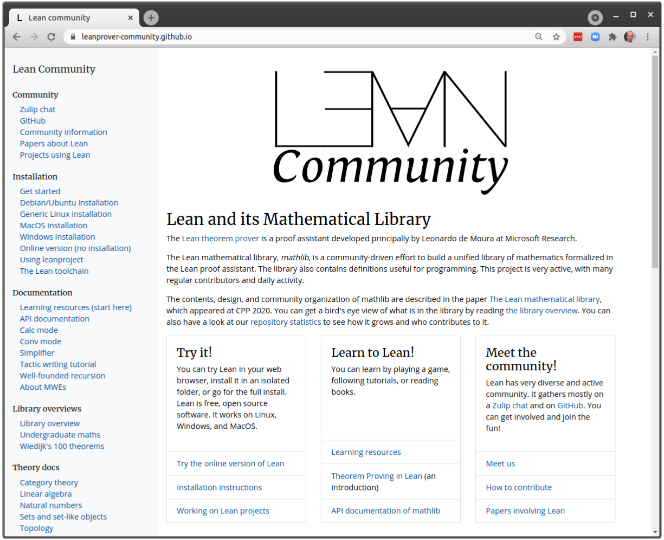
New technology for mathematics

The technology is based on *formal methods* in computer science, namely, logic-based computational methods for specifying and verifying software and hardware.

Until recently, very few mathematicians were using proof assistants.

In 2017, a number of mathematicians discovered Lean, and the Lean community was born.

The Lean community

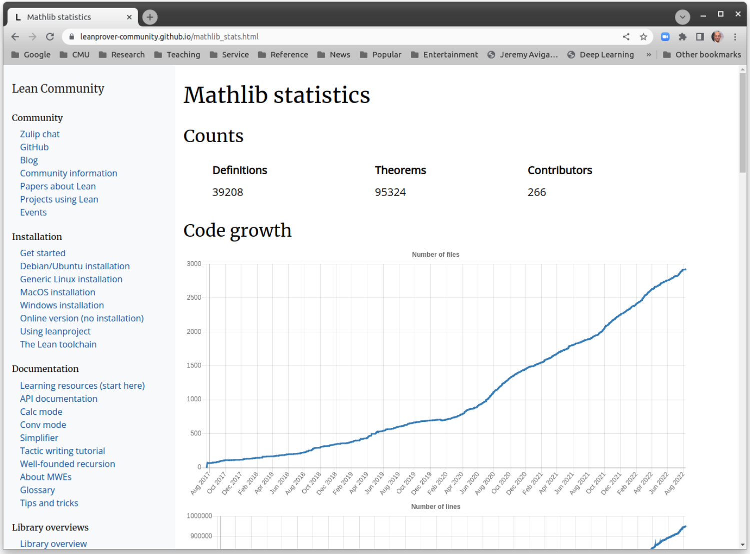


The Lean community

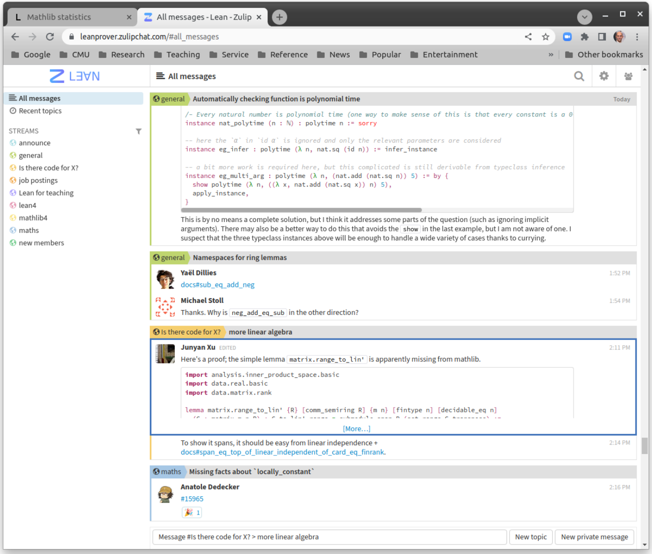
Where we are now:

- Hundreds of people have contributed to Lean's library, *mathlib*.
- The library has almost a million lines of formal definitions, theorems, and proofs.
- Lean's social media channel on *Zulip* gets hundreds of messages every day.
- There are a growing number of papers, conferences, and workshops dedicated to formalization of mathematics and Lean.

Mathlib statistics



The Lean Zulip channel



Notable achievements

For example:

- Jesse Han and Floris van Doorn gave the first formal verification of the independence of the continuum hypothesis, an important result in set theory.
- Johan Commelin led the *Liquid Tensor Experiment*, in response to a challenge by Fields Medalist Peter Scholze.
- Bhavik Mehta and Thomas Bloom verified an important result in number theory.

Scholze: "I find it absolutely insane that interactive proof assistants are now at the level that within a very reasonable time span they can formally verify difficult original research."

Lean in the news

Lean has been getting good press:

- *Quanta:* "Building the mathematical library of the future"
- *Quanta:* "At the Math Olympiad, computers prepare to go for the gold"
- *Nature:* "Mathematicians welcome computer-assisted proof in 'grand unification' theory"
- *Quanta:* "Proof Assistant Makes Jump to Big-League Math"

Kevin Buzzard recently gave a talk, "The Rise of Formalism in Mathematics," at the 2022 International Congress of Mathematicians.

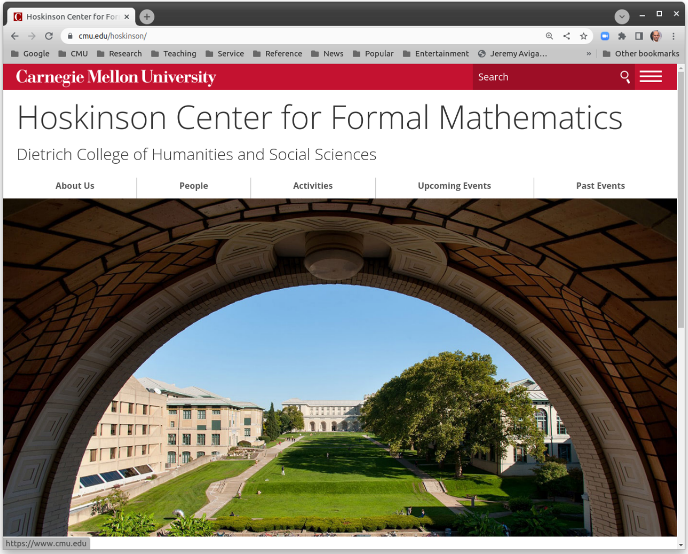
The Hoskinson Center

In September of 2021, Carnegie Mellon launched the Charles C. Hoskinson Center for Formal Mathematics.

It is dedicated to the use of formal computational methods in mathematical research and education.

Most of the center's activities are based on Lean.

The Hoskinson Center



What's the big deal?

Some are calling this the start of a revolution in mathematics.

It's reasonable to ask: why all the excitement?

This talk:

- Lean and formal methods in mathematics
- the nature of mathematical revolutions
- the digital revolution in mathematics

Revolutions in mathematics

Examples:

- the appearance of deductive reasoning in ancient Greece.
- the rise of algebraic methods
- the birth of calculus
- the inauguration of infinitary reasoning in the 19th century
- the advent of the computer and numeric computation

The rise of algebraic methods

The roots of algebra can be found in Al-Khwarizmi (9th century), and even earlier in ancient Greece.

The turning point in the early 17th century:

- the development of better algebraic notation
- the mathematization of natural science
- the use of algebraic methods to solve problems in geometry and science

The rise of algebraic methods

Cardano's solution $x^3 + px = q$ in 1545:

"Cube the third part of the number of unknowns, to which you add the square of half the number of the equation, and take the root of the whole, that is, the square root, which you will use, in one case adding the half of the number which you just multiplied by itself, in the other case subtracting the same half, and you will have a binomial and apotome respectively; then subtract the cube root of the apotome from the cube root of the binomial, and the remainder from this is the value of the unknown."

Today:

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

The rise of algebraic methods

Galileo, *The Assayer*, 1623:

"Philosophy [i.e. natural philosophy] is written in this grand book — I mean the Universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics..."

The rise of algebraic methods

1637:

The Geometry of René Descartes

BOOK I

Problems the Construction of Which Requires Only Straight Lines and Circles

ANY problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.[1] Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division and the extraction of roots, which may be considered a kind of division, so in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call unity in order to

The rise of algebraic methods

Often it is not necessary thus to draw the lines on paper, but it is sufficient to designate each by a single letter. Thus, to add the lines BD and GH, I call one a and the other b, and write $a + b$. Then $a - b$ will indicate that b is subtracted from a; ab that a is multiplied by b; $\frac{a}{b}$ that a is divided by b; aa or a^2 that a is multiplied by itself; a^3 that this result is multiplied by a, and so on, indefinitely.[4] Again, if I wish to extract the square root of a^2+b^2, I write $\sqrt{a^2+b^2}$; if I wish to extract the cube root of $a^3-b^3+ab^2$, I write $\sqrt[3]{a^3-b^3+ab^2}$, and similarly for other roots.[5] Here it must be observed that by a^2, b^3, and similar expressions, I ordinarily mean only simple lines, which, however, I name squares, cubes, etc., so that I may make use of the terms employed in algebra.[6]

Mathematical revolutions

Mathematical revolutions don't happen all at once.

They are not revolutions in the sense of overthrowing the old order. Rather they incorporate the past and build on it.

They open up new capacities for thought:

- Things that were hard become easier.
- Problems that were out of reach become solvable.
- New questions and problems arise.

What's important to mathematics

- Mathematics has practical applications.
- What we really care about in mathematical *understanding*.
- We value powerful intuitions, insights, and ideas.
- We need to communicate these ideas to one another in precise ways.
- The subject provides extraordinary means to come to consensus as to whether a proof is correct.
- The main challenge is complexity.

The digital revolution

Remember the outline:

- Lean and formal methods in mathematics
- the nature of mathematical revolutions
- the digital revolution in mathematics

Let's think about the new technology in these terms.

Verifying correctness

In early 2022, Thomas Bloom solved a problem posed by Paul Erdős and Ronald Graham.

The headline in Quanta read "Math's 'Oldest Problem Ever' Gets a New Answer."

Within in a few months, Bloom and Bhavik Mehta verified the correctness of the proof in Lean.

Verifying correctness

Timothy Gowers
@wtgowers · Jun 13

Very excited that Thomas Bloom and Bhavik Mehta have done this. I think it's the first time that a serious contemporary result in "mainstream" mathematics doesn't have to be checked by a referee, because it has been checked formally. Maybe the sign of things to come ... 1/

> **Kevin Buzzard** @XenaProject · Jun 12
>
> Happy to report that Bloom went on to learn Lean this year and, together with Bhavik Mehta, has now formalised his proof in Lean b-mehta.github.io/unit-fractions/ (including formalising the Hardy-Littlewood circle method), finishing before he got a referee's report for the paper ;-)
>
> Show this thread

2 26 141

Exploring mathematics

Similarly, Peter Scholze wrote:

"I am excited to announce that the Experiment has verified the entire part of the argument that I was unsure about."

But he went on:

"[H]alf a year ago, I did not understand why the argument worked. . . ."

The formalization helped him realize that

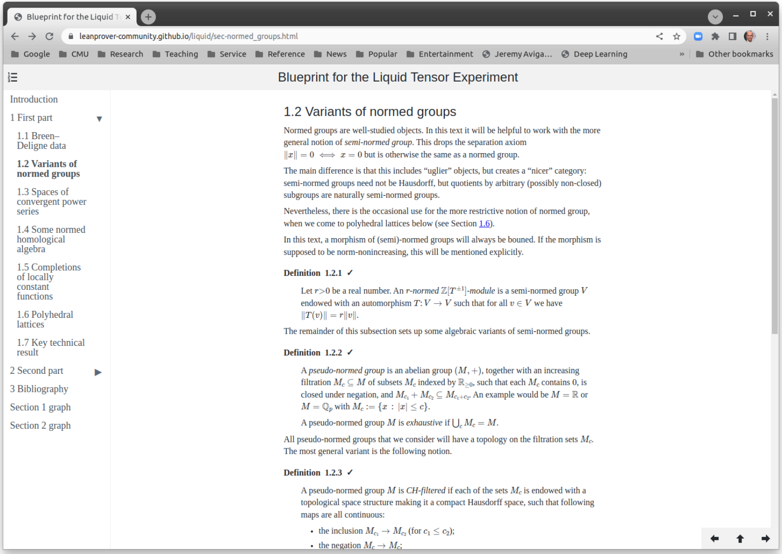
"the key thing happening is a reduction from a non-convex problem over the reals to a convex problem over the integers."

Collaboration

The liquid tensor experiment is also a model for digital collaboration.

- The formalization was in kept in a shared online repository.
- Participants followed an informal blueprint with links to the repository.
- Participants were in constant contact on Zulip.
- Lean made sure the pieces fit together.

Collaboration



Blueprint for the Liquid Tensor Experiment

Introduction
1 First part
1.1 Breen–Deligne data
1.2 Variants of normed groups
1.3 Spaces of convergent power series
1.4 Some normed homological algebra
1.5 Completions of locally constant functions
1.6 Polyhedral lattices
1.7 Key technical result
2 Second part
3 Bibliography
Section 1 graph
Section 2 graph

1.2 Variants of normed groups

Normed groups are well-studied objects. In this text it will be helpful to work with the more general notion of *semi-normed group*. This drops the separation axiom $\|x\| = 0 \iff x = 0$ but is otherwise the same as a normed group.

The main difference is that this includes “uglier” objects, but creates a “nicer” category: semi-normed groups need not be Hausdorff, but quotients by arbitrary (possibly non-closed) subgroups are naturally semi-normed groups.

Nevertheless, there is the occasional use for the more restrictive notion of normed group, when we come to polyhedral lattices below (see Section 1.6).

In this text, a morphism of (semi)-normed groups will always be bouned. If the morphism is supposed to be norm-nonincreasing, this will be mentioned explicitly.

Definition 1.2.1 ✓

Let $r>0$ be a real number. An *r-normed $\mathbb{Z}[T^{\pm 1}]$-module* is a semi-normed group V endowed with an automorphism $T: V \to V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

The remainder of this subsection sets up some algebraic variants of semi-normed groups.

Definition 1.2.2 ✓

A *pseudo-normed group* is an abelian group $(M, +)$, together with an increasing filtration $M_c \subseteq M$ of subsets M_c indexed by $\mathbb{R}_{\geq 0}$, such that each M_c contains 0, is closed under negation, and $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$. An example would be $M = \mathbb{R}$ or $M = \mathbb{Q}_p$ with $M_c := \{x : |x| \leq c\}$.

A pseudo-normed group M is *exhaustive* if $\bigcup_c M_c = M$.

All pseudo-normed groups that we consider will have a topology on the filtration sets M_c. The most general variant is the following notion.

Definition 1.2.3 ✓

A pseudo-normed group M is *CH-filtered* if each of the sets M_c is endowed with a topological space structure making it a compact Hausdorff space, such that following maps are all continuous:

- the inclusion $M_{c_1} \to M_{c_2}$ (for $c_1 \leq c_2$);
- the negation $M_c \to M_c$;

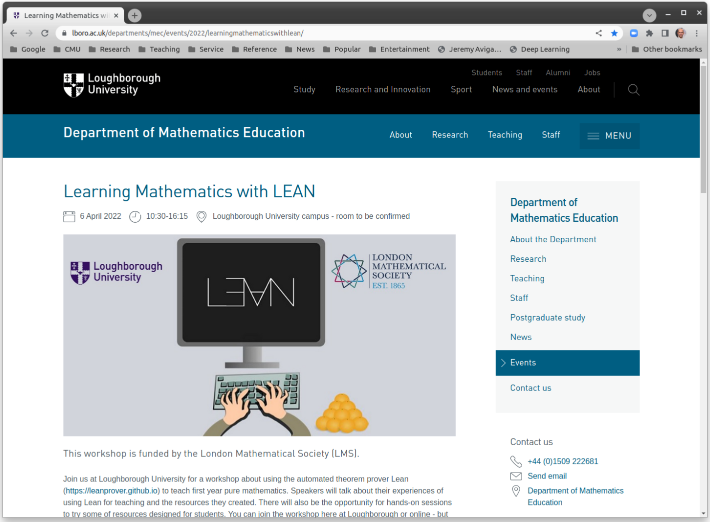
Teaching

An interactive proof assistant is a powerful tool for teaching mathematics.

It empowers students to explore mathematical reasoning on their own.

There have been workshops and conference sessions dedicated to learning how to use the technology effectively.

Teaching



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Learning Mathematics with LEAN
6 April 2022
10:30-16:15
Loughborough University campus - room to be confirmed
Loughborough University
LEAN
LONDON MATHEMATICAL SOCIETY EST. 1865
This workshop is funded by the London Mathematical Society (LMS).
Join us at Loughborough University for a workshop about using the automated theorem prover Lean (https://leanprover.github.io) to teach first year pure mathematics. Speakers will talk about their experiences of using Lean for teaching and the resources they created. There will also be the opportunity for hands-on sessions to try some of resources designed for students. You can join the workshop here at Loughborough or online - but
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Other applications

This only scratches the surface.

Lean can also be used as a platform for numerical and symbolic computation, as well as automated reasoning and machine learning.

- It enables us to apply computational tools to precise mathematical formulations.
- It can be used to verify and interpret the computational results.

The digital revolution in mathematics

Formal technology can help us:

- build mathematical libraries,
- verify results,
- explore new concepts,
- collaborate,
- teach mathematics,
- carry out mathematical computation more rigorously, and
- use AI to discover new mathematics.

Revolutions in mathematics

I sometimes wonder whether people *knew*, at the time, that they were in middle of a revolution.

Projectile motion

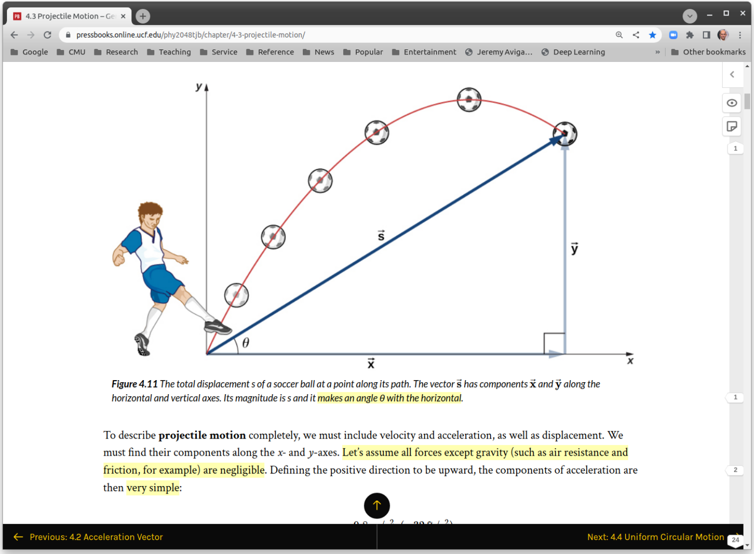

***Figure 4.11** The total displacement s of a soccer ball at a point along its path. The vector $\vec{s}$ has components $\vec{x}$ and $\vec{y}$ along the horizontal and vertical axes. Its magnitude is s and it makes an angle θ with the horizontal.*

To describe **projectile motion** completely, we must include velocity and acceleration, as well as displacement. We must find their components along the *x*- and *y*-axes. Let's assume all forces except gravity (such as air resistance and friction, for example) are negligible. Defining the positive direction to be upward, the components of acceleration are then very simple:

Projectile motion

Horizontal Motion

$$v_{0x} = v_x,\ x = x_0 + v_x t$$

Vertical Motion

$$y = y_0 + \tfrac{1}{2}(v_{0y} + v_y)t$$

$$v_y = v_{0y} - gt$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2$$

$$v_y^2 = v_{0y}^2 - 2g(y - y_0)$$

Using this set of equations, we can analyze projectile motion, keeping in mind some important points.

Problem-Solving Strategy: Projectile Motion

1. Resolve the motion into horizontal and vertical components along the *x*– and *y*-axes. The magnitudes of the components of displacement $\vec{s}$ along these axes are *x* and *y*. The magnitudes of the components of velocity $\vec{v}$ are $v_x = v\cos\theta$ and $v_y = v\sin\theta$, where v is the magnitude of the velocity and θ is its direction relative to the horizontal, as shown in Figure.
2. Treat the motion as two independent one-dimensional motions: one horizontal and the other vertical. Use the kinematic equations for horizontal and vertical moti esented earlier.
3. Solve for the unknowns in the two separate motions: one horizontal and one vertical. Note that the only com-

Projectile motion

THIRD DAY

[190]

CHANGE OF POSITION. [*De Motu Locali*]

MY purpose is to set forth a very new science dealing with a very ancient subject. There is, in nature, perhaps nothing older than motion, concerning which the books written by philosophers are neither few nor small; nevertheless I have discovered by experiment some properties of it which are worth knowing and which have not hitherto been either observed or demonstrated. Some superficial observations have been made, as, for instance, that the free motion [*naturalem motum*] of a heavy falling body is continuously accelerated;* but to just what extent this acceleration occurs has not yet been announced; for so far as I know, no one has yet pointed out that the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity.†

It has been observed that missiles and projectiles describe a curved path of some sort; however no one has pointed out the fact that this path is a parabola. But this and other facts, not few in number or less worth knowing, I have succeeded in proving; and what I consider more important, there have been opened up to this vast and most excellent science, of which my

Projectile motion

THEOREM I, PROPOSITION I

A projectile which is carried by a uniform horizontal motion compounded with a naturally accelerated vertical motion describes a path which is a semi-parabola.

Revolutions in mathematics

But these and other facts, not few in number or less worth knowing, I have succeeded in proving; and what I consider more important, there have been opened up to this vast and most excellent science, of which my work is merely the beginning, ways and means by which other minds more acute than mine will explore its remote corners.

Revolutions in mathematics

So that we may say the door is now opened, for the first time, to a new method fraught with numerous and wonderful results which in future years will command the attention of other minds.

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Thank you

https://www.andrew.cmu.edu/user/avigad