Some thoughts on metamathematics and proof theory

Jeremy Avigad

June 27, 2001

Below are two excerpts from a research statement I wrote recently (for the purposes of promotion within Carnegie Mellon).

Mathematical logic can be described as the mathematical study of the principles of reasoning. As such, the subject can be viewed as encompassing many different *kinds* of reasoning: mathematical/deductive, scientific/inductive, modal, probabilistic, nonmonotonic, temporal, and so on. In my research, I have focused on the mathematical analysis of mathematical reasoning itself, a branch of the subject that used to be called "metamathematics." (I have recently begun to grow fond of the word.) In the Hilbert tradition, I have favored a proof-theoretic approach, whereby mathematical practice is modeled by formal axiomatic deductive systems, and the metamathematical analysis is couched in the evaluation of their strengths and limitations. This practice brings benefits on a number of fronts:

1. *Philosophical*: While formal deductive systems shed light on epistemological issues (we come to know mathematical truths by proving them), they also help clarify various ontological stances by endowing them with concrete, formal representations. The metamathematical stance allows one to address methodological issues as well. For example, set theory has clarified the role that the axiom of choice and the continuum hypothesis play in ordinary mathematics; and modern proof theory offers refined analyses of forms of induction, compactness, choice principles, and types of constructions that are central to mathematical practice.

2. *Mathematical*: There is no sharp line to be drawn between mathematical activity and formal reflection on that activity. Metamathematical analysis contributes to mathematics by clarifying the fundamental

language, methods, and concepts involved; in addition, it can raise mathematical questions of its own, like questions of independence or decidability.

3. *Computational*: Of course, metamathematical considerations are important to automated deduction; moreover, they have helped guide the development of programming languages, and formal methods are increasingly used in hardware and software verification.

Proof theory was born of a particular tension in mathematics, which became salient in the wake of the Dedekind-Cantor revolution of the late nineteenth century. The new methods that had been introduced represented a shift from thinking of mathematics as a science of explicit computational reasoning about concretely presented objects (closely tied to their symbolic representations) to thinking of mathematics as a science of conceptual reasoning about abstractly presented objects, which stand apart from their symbolic representations. At stake was not just the *correctness* or *certainty* of the new methods, but, moreover, their *appropriateness* to mathematics. With Erich Reck, I have argued that Hilbert's program should not be viewed as just a failed attempt at providing epistemic certainty via consistency proofs, but rather, more positively, as a largely successful means of reconciling abstract and concrete views of mathematics via formal metamathematical analysis. This theme — exploring concrete, constructive, symbolic, and computational aspects of abstract mathematical reasoning — has driven much of my research.

For institutional reasons, research in metamathematics and proof theory is currently in danger of getting "squeezed out." Although mathematicians are generally interested in foundational and methodological issues, with today's problem-solving ethic in mathematics such reflective work is often viewed as being more appropriate to a philosophy department. Conversely, while philosophers are often respectful of mathematical logic's philosophical contributions, philosophers tend to view such work as properly mathematical. With recent applications to automated deduction, functional programming, and hardware and software verification, metamathematics and proof theory find some support in computer science departments; but, of course, this imposes particular computational orientations and standards on the research.

Ultimately, if the subject is to remain strong, it will have to prove its merits on each front independently. Nonetheless, there are good reasons for keeping interdisciplinary communication alive. For one thing, there is strength in numbers; the subject is healthier viewed as a single multifaceted discipline than as a handful of isolated minorities. Furthermore, ideas, methods, questions, and research programs developed in one branch are often interesting or relevant to another. Finally, for many researchers, it is exactly the interdisciplinary nature of the subject that attracted them to the subject in the first place, and it is the interplay of ideas that makes the subject so attractive.