An Effective Proof that Open Sets are Ramsey

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Abstract

Solovay has shown that if $\mathcal{O}$ is an open subset of $P(\omega)$ with code S and no infinite set avoids $\mathcal{O}$, then there is an infinite set hyperarithmetic in S that lands in $\mathcal{O}$. We provide a direct proof of this theorem that is easily formalizable in ATR_0.

1 Introduction

A plausible generalization of Ramsey's theorem asserts that for every two-coloring of the infinite subsets of ω there is an infinite homogeneous set, that is, an infinite subset of ω every infinite subset of which has been assigned the same color. Unfortunately, under the axiom of choice, this generalization is false: by transfinite recursion along a well-ordering of the reals one can cook up a coloring with no infinite homogeneous set. On the other hand, the nonconstructive nature of this counterexample suggests that perhaps the theorem might hold true for colorings that are "well-behaved" or "easily definable."

To that end, we define a **partition** to be a subset of the power set of ω, with the understanding that the infinite subsets falling inside the partition are colored, say, red, and those outside the partition are colored blue. If $\mathcal{P}$ is a partition and X is an infinite subset of ω, then X **lands in** $\mathcal{P}$ if every infinite subset of X is in $\mathcal{P}$, and X **avoids** $\mathcal{P}$ if no infinite subset of X is in $\mathcal{P}$. A partition $\mathcal{P}$ is **Ramsey** if there is an infinite set X that either lands in $\mathcal{P}$ or avoids $\mathcal{P}$. The theorems we are interested in are of the form "every well-behaved partition is Ramsey." A number of authors have shown independently that if $\mathcal{P}$ is open in the usual topology then it is Ramsey (see [4]), and the conclusion has been extended to Borel sets by Galvin and Prikry [4] and analytic sets by Silver [8, 2].

Solovay [11] has strengthened the result for open sets as follows: if $\mathcal{O}$ is an open set with code S and no infinite set avoids $\mathcal{O}$, then there is an infinite set hyperarithmetic in S which lands in $\mathcal{O}$. Mansfield [7] has provided a shorter proof of this theorem that was used in [3] to show that the subsystem of second-order arithmetic ATR_0 proves (and is in fact over a weak base theory equivalent to) Solovay's result. The formalization of Mansfield's proof in ATR_0 is, however, somewhat difficult.

Below we present a remarkably direct proof of Solovay's theorem, obtained by "effectivizing" an argument that uses a nonprincipal ultrafilter on ω. Our proof is easily formalizable in ATR_0. For more elaborate uses of ultrafilter methods in proving Ramsey-theoretic statements see [6, 5, 1], and for more information on ATR_0 and other subsystems of second-order arithmetic see, for example, [3, 9, 10].

I'd like to thank Andreas Blass for showing me the ultrafilter proof in Section 2 and suggesting the use of Lemma 3.2, and Stephen Simpson for helpful comments on a draft of this paper. The effective proof of Solovay's theorem appears in Section 3.

2 The noneffective version

From now on we identify finite and infinite subsets of ω with the sequences that enumerate their elements in increasing order. Let T be the tree of finite increasing sequences from ω, and let the variables α, β, σ, τ denote elements of T. The notation $\sigma \subseteq \tau$ means that (the set associated with) σ is a subset of (the set associated with) τ and not necessarily that σ is an initial segment of τ.

A basis for the usual topology on $P(\omega)$ is given by sets of the form

$$\mathcal{B}_\sigma = \{X \mid X \text{ extends } \sigma\},$$

and a set of sequences S can be taken to code the open set

$$\mathcal{O} = \bigcup_{\sigma \in S} \mathcal{B}_\sigma.$$

Though the assignment of codes to open sets is not unique, it is well known that a set $\mathcal{O}$ is Σ^0_1 definable from a parameter A if and only if $\mathcal{O}$ is open and has a code recursive in A.

Theorem 2.1 *Open sets are Ramsey.*

Proof. Let $\mathcal{O}$ be an open subset of $P(\omega)$ with code S. Without loss of generality we can assume that S is closed under extensions, since otherwise the set

$$S' = \{\sigma \mid \text{some initial segment of } \sigma \text{ is in } S\}$$

also codes $\mathcal{O}$ and has this property. Fix $\mathcal{U}$, a nonprincipal ultrafilter on ω.

By transfinite recursion on the ordinals we label certain elements σ of T **good** and associate an element U_σ of $\mathcal{U}$. At stage 0, we label a sequence σ good if σ is in S, and set $U_\sigma = \omega$. At stage μ we label σ good if σ has not already been so labelled and the set of elements n such that $\sigma\hat{}n$ is good is in $\mathcal{U}$. In this case we set

$$U_\sigma = \{n \mid \sigma\hat{}n \text{ was labelled good before stage } \mu\}.$$

Since T is countable, this process stabilizes at some stage before ω_1. At this point label the remaining elements σ of T **bad** and set

$$U_\sigma = \{n \mid \sigma\hat{}n \text{ is bad}\}.$$

Note that if σ is bad then U_σ is in $\mathcal{U}$, since otherwise its complement would be in $\mathcal{U}$ and we would have labelled σ good.

We claim that if the empty sequence is bad, there is a set which avoids $\mathcal{O}$, and if empty sequence is good, there is a set which lands in $\mathcal{O}$.

Suppose the empty sequence is bad. We construct an increasing sequence $x_0, x_1, x_2, \ldots$ every subsequence of which is bad. Take x_0 to be any element of $U_{\langle\rangle}$. Once $x_0, x_1, \ldots, x_n$ have been chosen, note that the set

$$\bigcap_{\sigma \subseteq \langle x_0, x_1, \ldots, x_n\rangle} U_\sigma$$

is in $\mathcal{U}$, and so we can take x_{n+1} to be any element of this set that is greater than x_n.

Let $X = \langle x_0, x_1, x_2, \ldots\rangle$. This set X avoids $\mathcal{O}$: if some $Y \subseteq X$ were an element of $\mathcal{O}$, we'd have a sequence $\langle y_0, y_1, \ldots, y_n\rangle \subseteq X$ in S. But this sequence would have been labelled good at stage 0, contradicting the fact that every subsequence of X is bad.

So now suppose the empty sequence is good. Exactly as before, construct an increasing sequence $x_0, x_1, x_2, \ldots$ every subsequence of which is good. Let $X = \langle x_0, x_1, x_2, \ldots\rangle$. We claim that X lands in $\mathcal{O}$. Let $Y = \langle y_0, y_1, y_2, \ldots\rangle$ be any infinite subset of X, and for each n let μ_n be the stage at which $\langle y_0, y_1, \ldots, y_n\rangle$ was labelled good. Then if $\mu_n \neq 0$ we have that $\mu_{n+1} < \mu_n$, since y_{n+1} is in $U_{\langle y_0, y_1, \ldots, y_n\rangle}$ and $\langle y_0, y_1, \ldots, y_n\rangle$ was labelled good by virtue of this set. Since any descending sequence of ordinals must eventually hit 0, we will have $\mu_m = 0$ for some m, in which case $\langle y_0, y_1, \ldots, y_m\rangle \in S$ and hence $Y \in \mathcal{O}$. □

3 The effective version

Making the foregoing argument more effective involves two observations:

1. We don't need the entire ultrafilter $\mathcal{U}$; it is enough to keep track of countably many sets that we've committed to being in the ultrafilter.

2. We don't need the entire tree T. On the assumption that no set avoids $\mathcal{O}$, we can restrict our attention to a well-founded subtree T', and then label the nodes "from the bottom up."

Theorem 3.1 *Let $\mathcal{O}$ be an open subset of $P(\omega)$ with code S, and suppose no infinite X avoids $\mathcal{O}$. Then there is an infinite X hyperarithmetic in S, such that X lands in $\mathcal{O}$.*

Proof. Fix $\mathcal{O}$ and S as in the hypothesis of the theorem, and suppose no infinite X avoids $\mathcal{O}$. Let

$$T' = \{\sigma \mid \text{no subsequence of } \sigma \text{ is in } S\}$$

and note that T' is a tree that is closed under subsequences. We claim T' is well-founded: Since no infinite X avoids $\mathcal{O}$, every infinite X has a finite subsequence σ in S. But no such X can be a path through T'.

We start by labelling sequences outside of T' either good or bad. If σ is outside of T', let τ be the smallest initial segment of σ that is outside of T'. If τ is in S we label σ good, and otherwise we label σ bad.

Recall the Brouwer-Kleene ordering on T', in which $\sigma \prec \tau$ iff σ *extends* τ or σ is less than τ in the lexicographical ordering. Since T' is well-founded, $\prec$ is a well-ordering. Our construction proceeds by transfinite recursion along $\prec$, where at stage α we label the node α good or bad and at the same time define a set U_α, so that the following hold:

1. Each U_α is infinite.
2. If $\alpha \succ \beta$ then $U_\alpha \subseteq_f U_\beta$, i.e. $U_\alpha \setminus U_\beta$ is finite.
3. If α is good then for all $n \in U_\alpha$, $\alpha\hat{\ }n$ is good.
4. If α is bad then for all $n \in U_\alpha$, $\alpha\hat{\ }n$ is bad.

We will need to use the following

Lemma 3.2 *Suppose for each $\beta \prec \alpha$ we've chosen U_β so that clauses (1) and (2) hold. Then there is an infinite set Z such that for every $\beta \prec \alpha$ we have $Z \subseteq_f U_\beta$.*

Proof. If α is the least element in the ordering we can take $Z = \omega$, and if α is the successor of β we can take $Z = U_\beta$. In the case where α is a limit, we take a diagonal intersection: since there are only countably many $\beta \prec \alpha$ we can find a countable sequence β_i cofinal in α. Take u_0 to be the least element in U_{β_0}, and take u_{i+1} to be the least element in

$$\bigcap_{j \le i} (U_{\beta_j} \setminus \{u_j\}).$$

It is straightforward to verify that $Z = \{u_0, u_1, u_2 \ldots\}$ has the desired property. □

We now describe the construction. Suppose we've constructed U_β for all $\beta \prec \alpha$ and labelled each node $\beta \prec \alpha$ good or bad, so that clauses (1)-(4) hold.

At stage α, first use the lemma to pick an infinite Z so that for all $\beta \prec \alpha$, $Z \subseteq_f U_\beta$. Then consider

$$W = \{n \in Z \mid \sigma\hat{\ }n \text{ is good}\}.$$

If W is infinite, label α good and take $U_\alpha = W$. Otherwise label α bad and take $U_\alpha = Z \setminus W$. The process continues until the empty sequence (i.e. the root of T) has been labelled and $U_{\langle\rangle}$ has been defined.

Now define $U_\sigma = \omega$ for all σ outside of T', and note that clauses (3) and (4) still hold for such σ.

We claim that the empty sequence is good. To prove the claim, suppose the empty sequence were bad. We build an increasing sequence of elements $x_0, x_1, x_2, \ldots$, every subsequence of which is bad. Let x_0 be any element of $U_{\langle\rangle}$ and once $x_0, x_1, \ldots, x_n$ have been chosen, let

$$U = \bigcap_{\sigma \subseteq \langle x_0, \ldots, x_n \rangle} U_\sigma.$$

Since $U_{\langle\rangle} \subseteq_f U_\sigma$ for each of these (finitely many) σ, we have $U_{\langle\rangle} \subseteq_f U$, and hence U is infinite. Take x_{n+1} to be any (e.g. the least) element of U that is greater than x_n.

Let $X = \{x_0, x_1, x_2, \ldots\}$. Since we're assuming that no infinite set avoids $\mathcal{O}$, some subsequence σ of X is in S. Take σ minimal, so that no proper subsequence of σ is in S. Then σ is outside of T' and every initial segment of σ is in T'. But we initially labelled such σ good, contradiction. This proves our claim that the empty sequence is good.

Now use the same construction to obtain an increasing sequence $x_0, x_1, x_2, \ldots$ every subsequence of which is good. Let $X = \{x_0, x_1, x_2, \ldots\}$. We claim that X lands in O. Let Y be any infinite subset of X. Since T' is well-founded, there is a smallest initial segment σ of Y that is outside of T. By our construction of X we know that σ is good, and hence σ is in S. So Y is in O, proving our claim.

Since the ordering $\prec$ is recursive in S, and for each σ in T' the set U_σ is arithmetically definable from S and the sequence $\langle U_\tau \rangle_{\tau \prec \sigma}$, it is easy to verify that X is hyperarithmetic in S. □

Corollary 3.3 *ATR_0 proves Theorem 3.1 (and hence the fact that open sets are Ramsey).*

Proof. Formalizing the above argument in ATR_0 is straightforward (see [3, 9, 10]). □

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