Definition FS.5.1: If x,$y \in \omega$ and $y \neq 0$ then x / y = (x,y). Otherwise x / y is undefined. Precedence: 5.

Definition FS.5.2: *The set of positive fractions* is the set of x / y such that x / y is defined.

Definition FS.5.3: $x \equiv y$ if and only if there exist a, b, c, d such that $x = a$ / b and $y = c$ / d and $a \times d = b \times c$.

Definition FS.5.4: $x < y$ if and only if there exist a, b, c, d such that $x = a$ / b and $y = c$ / d and $a \times d < b \times c$.

Definition FS.5.5: $x > y$ if and only if $y < x$.

Definition FS.5.6: $x \leq y$ if and only if $x < y$ or $x \equiv y$.

Definition FS.5.7: $x \geq y$ if and only if $x > y$ or $x \equiv y$.

Definition FS.5.8: $x + y$ is the unique z such that there exist a, b, c, d, e, f such that $x = a$ / b and $y = c$ / d and $z = e$ / f and $e = a \times d + b \times c$ and $f = b \times d$. Precedence: 40.

Definition FS.5.9: $x \times y$ is the unique t such that there exist a, b, c, d, e, f such that $x = a$ / b and $y = c$ / d and $t = e$ / f and $e = a \times c$ and $f = b \times d$. Precedence: 20.

Definition FS.5.10: *The set Nra* is the set of the coset of x with respect to $\{(u, v) : u \equiv v\}$ such that x is in the set of positive fractions.

Definition FS.5.10.5: If x is in the set of positive fractions then x is the coset of x with respect to $\{(u, v) : u \equiv v\}$. Otherwise x is undefined.

Definition FS.5.11: $x < y$ if and only if x, y are in the set Nra and there exist u, v such that $u \in x$ and $v \in y$ and $u < v$.

Definition FS.5.12: $x > y$ if and only if x, y are in the set Nra and there exist u, v such that $u \in x$ and $v \in y$ and $u > v$.

Definition FS.5.13: $x \leq y$ if and only if x, y are in the set Nra and there exist u, v such that $u \in x$ and $v \in y$ and $u \leq v$.

Definition FS.5.14: $x \geq y$ if and only if x, y are in the set Nra and there exist u, v such that $u \in x$ and $v \in y$ and $u \geq v$.

Definition FS.5.15: $x + y$ is the unique z such that x, y, z are in the set Nra and there exist u, v, w such that $u \in x$ and $v \in y$ and $w \in z$ and $u + v \equiv w$. Precedence: 40.

Definition FS.5.16: $x \times y$ is the unique z such that x, y, z are in the set Nra and there exist u, v, w such that $u \in x$ and $v \in y$ and $w \in z$ and $u \times v \equiv w$. Precedence: 20.

Definition FS.5.17: 0 is the coset of 0 / 1 with respect to $\{(x, y) : x \equiv y\}$.

Definition FS.5.18: 1 is the coset of 1 / 1 with respect to $\{(x, y) : x \equiv y\}$.

Definition FS.5.19: $x \equiv y$ if and only if there exist a, b, c, d such that x = (a,b) and y = (c,d) and $a + d = b + c$.

Definition FS.5.20: $x < y$ if and only if there exist a, b, c, d such that x = (a,b) and y = (c,d) and $a + d < b + c$.

Definition FS.5.21: $x + y$ is the unique z such that there exist a, b, c, d, e, f such that x = (a,b) and y = (c,d) and z = (e,f) and $a + c + f = b + d + e$. Precedence: 40.

Definition FS.5.22: $x \times y$ is the unique z such that there exist a, b, c, d, e, f such that x = (a,b) and y = (c,d) and z = (e,f) and $a \times c + b \times d + f = a \times d + b \times c + e$. Precedence: 20.

Definition FS.5.23: $\mathbb{Q}$ is the set of the coset of x with respect to $\{(u, v) : u \equiv v\}$ such that x is in the cartesian product of the set Nra and the set Nra.

Definition FS.5.23.5: If x is in the set Nra then x is the coset of (x,0) with respect to $\{(u, v) : u \equiv v\}$.

Definition FS.5.23.8: If x is in the set of positive fractions then $x = x$.

Definition FS.5.23.A: $\mathbb{Q} = \mathbb{Q}$.

Definition FS.5.24: $x < y$ if and only if there exist z, w such that x,$y \in \mathbb{Q}$ and $z \in x$ and $w \in y$ and $z < w$.

Definition FS.5.25: $x + y$ is the unique z such that x,y,$z \in \mathbb{Q}$ and there exist a, b, c such that $a \in x$ and $b \in y$ and $c \in z$ and $a + b = c$. Precedence: 40.

Definition FS.5.26: $x \times y$ is the unique z such that x,y,$z \in \mathbb{Q}$ and there exist a, b, c such that $a \in x$ and $b \in y$ and $c \in z$ and $a \times b = c$. Precedence: 20.

Definition FS.5.27: 0 is the coset of (0,0) with respect to $\{(x, y) : x \equiv y\}$.

Definition FS.5.28: 1 is the coset of (1,0) with respect to $\{(x, y) : x \equiv y\}$.

Definition FS.5.29: $x > y$ if and only if $y < x$.

Definition FS.5.30: $x \leq y$ if and only if $x < y$ or $x = y$.

Definition FS.5.31: $x \geq y$ if and only if $x > y$ or $x = y$.

Definition FS.5.32: $x - y = (!z)x = y + z$. Precedence: 60.

Definition FS.5.33: $|x|$ is the unique $y \in \mathbb{Q}$ such that if $x \geq 0$ then $y = x$ and if $x < 0$ then $y = 0 - x$.

Definition FS.5.35: $\mathbb{N}$ is the unique x such that for every y, $y \in x$ if and only if $y = 0$ or $y > 0$ and $y - 1 \in x$.

Definition FS.5.35.A: $\mathbb{N} = \mathbb{N}$.

Definition FS.5.36: $\mathbb{Z}$ is the unique x such that for every y, $y \in x$ if and only if $y \in \mathbb{N}$ or $0 - y \in \mathbb{N}$.

Definition FS.5.36.A: $\mathbb{Z} = \mathbb{Z}$.

Definition FS.5.37: *The set of all sequences of rational numbers* is the set of maps from ω to $\mathbb{Q}$.

Definition FS.5.38: $x + y$ is the unique z such that x, y, z are in the set of all sequences of rational numbers and for every $n \in \omega$, $z(n) = x(n) + y(n)$. Precedence: 40.

Definition FS.5.39: $x \times y$ is the unique z such that x, y, z are in the set of all sequences of rational numbers and for every $n \in \omega$, $z(n) = x(n) \times y(n)$. Precedence: 20.

Definition FS.5.40: $x < y$ if and only if x,$y \in \omega$ and $x < y$.

Definition FS.5.41: $x > y$ if and only if x,$y \in \omega$ and $x > y$.

Definition FS.5.42: $x \leq y$ if and only if $x < y$ or $x = y$.

Definition FS.5.43: $x \geq y$ if and only if $x > y$ or $x = y$.

Definition FS.5.44: *The set of Cauchy sequences of rational numbers* is the set of x in the set of all sequences of rational numbers such that for every $\varepsilon > 0$, there exists $n \in \omega$ such that for every m, $r > n$, $|x(m) - x(r)| < \varepsilon$.

Definition FS.5.45: $x \equiv y$ if and only if for every $\varepsilon > 0$, there exists $n \in \omega$ such that for every $m > n$, $|x(m) - y(m)| < \varepsilon$.

Definition FS.5.46: $x < y$ if and only if x, y are in the set of Cauchy sequences of rational numbers and there exists $\delta > 0$ such that there exists $n \in \omega$ such that for every $m > n$, $x(m) + \delta < y(m)$.

Definition FS.5.47: $\mathbb{R}$ is the set of the coset of x with respect to $\{(u, v) : u \equiv v\}$ such that x is in the set of Cauchy sequences of rational numbers.

Definition FS.5.48.1: $x < y$ if and only if there exist z, w such that x,y $\in \mathbb{R}$ and $z \in x$ and $w \in y$ and $z < w$.

Definition FS.5.48.2: $x > y$ if and only if $y < x$.

Definition FS.5.48.3: $x \leq y$ if and only if $x < y$ or $x = y$.

Definition FS.5.48.4: $x \geq y$ if and only if $x > y$ or $x = y$.

Definition FS.5.49: $x + y$ is the unique z such that x,y,$z \in \mathbb{R}$ and there exist a, b, c such that $a \in x$ and $b \in y$ and $c \in z$ and $a + b \equiv c$. Precedence: 40.

Definition FS.5.49.A: $x - y = (!z)x = y + z$. Precedence: 60.

Definition FS.5.50: $x \times y$ is the unique z such that x,y,$z \in \mathbb{R}$ and there exist a, b, c such that $a \in x$ and $b \in y$ and $c \in z$ and $a \times b \equiv c$. Precedence: 20.

Definition FS.5.51: 0 is the unique $x \in \mathbb{R}$ such that there exists $w \in x$ such that for every $n \in \omega$, $w(n) = 0$.

Definition FS.5.52: 1 is the unique $x \in \mathbb{R}$ such that there exists $w \in x$ such that for every $n \in \omega$, $w(n) = 1$.

Definition FS.5.53: $|x|$ is the unique $y \in \mathbb{R}$ such that if $x \geq 0$ then $y = x$ and if $x < 0$ then $y = 0 - x$.

Definition FS.5.53.A: *The identity function on* $\mathbb{R}$ is the unique $y \in \mathbb{R}$ such that there exists $w \in y$ such that for every $n \in \omega$, $w(n) = x$.

Definition FS.5.53.B: If x is in the set Nra then x is the identity function on $\mathbb{R}$.

Definition FS.5.53.C: If x is in the set of positive fractions then x is the identity function on $\mathbb{R}$.

Definition FS.5.54: $\mathbb{Q}$ is the set of the identity function on $\mathbb{R}$ such that x $\in \mathbb{Q}$.

Definition FS.5.55: *The set of sequences of real numbers* is the set of maps from ω to $\mathbb{R}$.

Definition FS.5.56: *The set of Cauchy sequences of real numbers* is the set of x in the set of sequences of real numbers such that for every $\varepsilon > 0$, there exists $n \in \omega$ such that for every m, $r > n$, $|x(m) - x(r)| < \varepsilon$.

Definition FS.5.57: If x is in the set of sequences of real numbers then $\lim x$ is the unique $y \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n \in \omega$ such that for every $m > n$, $|x(m) - y| < \varepsilon$.

Definition FS.5.58: x is an *upper bound* on A if and only if $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ and for every $y \in A$, $y \leq x$.

Definition FS.5.59: If $A \subseteq \mathbb{R}$ then *the minimal element of* A is the unique $x \in A$ such that for every $y \in A$, it is not the case that $y < x$.

Definition FS.5.59.5: If $A \subseteq \mathbb{R}$ then *the maximal element of* A is the unique $x \in A$ such that for every $y \in A$, it is not the case that $x < y$.

Definition FS.5.60: If $A \subseteq \mathbb{R}$ then *the least upper bound of* A is the minimal element of the set of x such that x is an upper bound on A.

Definition FS.5.61.pre: If there exists $n \in \omega$ such that f is a function from n to $\mathbb{R}$ then *the graph of the finite sum function* is the unique x such that for every $m \in \omega$, if $m < n$ then $x(0) = 0$ and x, evaluated at the successor of m equals $x(m)$ plus f, evaluated at the successor of m and if $m \geq n$ then x, evaluated at the successor of m equals 0 and for every m, $x(m)$ is defined if and only if $m \in \omega$.

Definition FS.5.61: If there exists $n \in \omega$ such that f is a function from n to $\mathbb{R}$ then $\sum_{k \in Dom(f)} f(k)$ is the unique $r \in \mathbb{R}$ such that (the domain of f,r) is in the graph of the finite sum function.

Definition FS.5.62: If $r \in \mathbb{R}$ then $\sqrt{r}$ is the unique $y \in \mathbb{R}$ such that $y \geq 0$ and $y \times y = r$.

Definition FS.5.63: $\sup A$ is the unique s such that s is a supremum for A, under $\{(x, y) : x < y\}$.

Definition FS.5.64: $\inf A$ is the unique g such that g is an infimum for A, under $\{(x, y) : x < y\}$.