Action Parameterization 1

Given:

\[ I_{\text{joint}} = I_{\text{COM}} + M \cdot LL^2 \]
\[ \tau = I_{\text{joint}} \ddot{\theta} \]
\[ x_{\text{hip}} = L \sin(\theta - \varphi) \]
\[ y_{\text{hip}} = L \cos(\theta - \varphi) \]
\[ \ddot{x}_{\text{COM}} = L \cos(\theta - \varphi) (\ddot{\theta} - \ddot{\varphi}) - L \cos(\theta - \varphi) (\dot{\theta} - \dot{\varphi})^2 + LL \cos(\theta) \ddot{\theta} - LL \sin(\theta) \dddot{\theta} \]
\[ \ddot{y}_{\text{COM}} = -L \sin(\theta - \varphi) (\ddot{\theta} - \ddot{\varphi}) - L \sin(\theta - \varphi) (\dot{\theta} - \dot{\varphi})^2 - LL \sin(\theta) \dddot{\theta} - LL \cos(\theta) \dddot{\theta} \]
\[ F_{x\text{COM}} = -F_l \sin(\theta - \varphi) + \frac{\tau \cos(\theta - \varphi)}{L} \]
\[ F_{y\text{COM}} = F_l \cos(\theta - \varphi) - Mg + \frac{\tau \sin(\theta - \varphi)}{L} \]
\[ F_{x\text{COM}} = M \ddot{x}_{\text{COM}} \]
\[ F_{y\text{COM}} = M \ddot{y}_{\text{COM}} \]

Note that we are assuming the change in leg length to be small, and that there is no reaction torque from the leg.

Rewrite \( x_{\text{hip}} \) and \( y_{\text{hip}} \):

\[ x_{\text{hip}} = x_{\text{COM}} - LL \sin(\theta) \]
\[ y_{\text{hip}} = y_{\text{COM}} - LL \cos(\theta) \]

Rewrite \( \varphi \):

\[ \varphi = \theta - \sin^{-1}\left(\frac{x_{\text{hip}}}{L}\right) = \theta - \cos^{-1}\left(\frac{y_{\text{hip}}}{L}\right) \]

Rewrite \( \ddot{x}_{\text{COM}} \):

\[ M \ddot{x}_{\text{COM}} = -F_l \sin(\theta - \varphi) + \frac{\tau \cos(\theta - \varphi)}{L} \]

Sub in \( \varphi \):

\[ \ddot{x}_{\text{COM}} = - \frac{F_l}{M} \left(\frac{x_{\text{COM}} - LL \sin(\theta)}{L}\right) + \frac{\tau}{ML} \left(\frac{y_{\text{COM}} - LL \cos(\theta)}{L}\right) \]

Similarly for \( \ddot{y}_{\text{COM}} \):

\[ \ddot{y}_{\text{COM}} = \frac{F_l}{M} \left(\frac{y_{\text{COM}} - LL \cos(\theta)}{L}\right) - g + \frac{\tau}{ML} \left(\frac{x_{\text{COM}} - LL \sin(\theta)}{L}\right) \]

Therefore, the equations of motion are:

\[ \ddot{x}_{\text{COM}} = - \frac{F_l}{M} \left(\frac{x_{\text{COM}} - LL \sin(\theta)}{L}\right) + \frac{\tau}{ML} \left(\frac{y_{\text{COM}} - LL \cos(\theta)}{L}\right) \]
\[ \ddot{y}_{\text{COM}} = \frac{F_i}{M} \left( \frac{y_{\text{COM}} - LL \cos(\theta)}{L} \right) - g + \frac{\tau}{ML} \left( \frac{x_{\text{COM}} - LL \sin(\theta)}{L} \right) \]

Choose our state to be:

\[ X = \begin{bmatrix} x_{\text{COM}} \\ \dot{x}_{\text{COM}} \\ y_{\text{COM}} \\ \dot{y}_{\text{COM}} \\ \theta \\ \dot{\theta} \end{bmatrix}, U = \begin{bmatrix} F_i \\ \tau \end{bmatrix}, Y = \begin{bmatrix} x_{\text{COM}} \\ y_{\text{COM}} \end{bmatrix} \]

State equations:

\[ \dot{X}_2 = -\frac{U_1}{M} \left( \frac{X_1 - LL \sin(X_5)}{L} \right) + \frac{U_2}{ML} \left( \frac{X_3 - LL \cos(X_5)}{L} \right) \]
\[ \dot{X}_3 = X_4 \]
\[ \dot{X}_4 = \frac{U_1}{M} \left( \frac{X_3 - LL \cos(X_5)}{L} \right) - g + \frac{U_2}{ML} \left( \frac{X_1 - LL \sin(X_5)}{L} \right) \]
\[ \dot{X}_5 = \frac{U_2}{l_{\text{joint}}} \]
\[ \dot{X}_6 = \frac{U_2}{l_{\text{joint}}} \]

Linearize about \((0, L + LL, 0)\) to get equilibrium \(X_e\) and \(U_e\), so:

\[ X_{1e} = 0 \]
\[ X_{3e} = L + LL \]
\[ X_{5e} = 0 \]

Setting \(\dot{X} = 0\), we get:

\[ 0 = X_{2e} \]
\[ 0 = X_{4e} \]
\[ 0 = X_{6e} \]
\[ 0 = U_{2e} \]

Also:

\[ 0 = -\frac{U_{1e}}{M} \left( \frac{0 - LL \sin(0)}{L} \right) + \frac{U_{2e}}{ML} \left( \frac{L + LL - LL \cos(0)}{L} \right) \]

So \(U_{2e} = 0\).

Similarly:

\[ 0 = \frac{U_{1e}}{M} \left( \frac{L + LL - LL \cos(0)}{L} \right) - g + \frac{0}{ML} \left( \frac{0 - LL \sin(0)}{L} \right) \]

So \(U_{1e} = Mg\).

We will use the following to get \(A, B, C, D\):
\[
A = \begin{bmatrix}
\frac{\partial f_1(X, U)}{\partial X_1} & \cdots & \frac{\partial f_1(X, U)}{\partial X_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n(X, U)}{\partial X_1} & \cdots & \frac{\partial f_n(X, U)}{\partial X_n}
\end{bmatrix}_{(X_0, U_0)}
\]

\[
B = \begin{bmatrix}
\frac{\partial f_1(X, U)}{\partial U_1} & \cdots & \frac{\partial f_1(X, U)}{\partial U_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n(X, U)}{\partial U_1} & \cdots & \frac{\partial f_n(X, U)}{\partial U_m}
\end{bmatrix}_{(X_0, U_0)}
\]

\[
C = \begin{bmatrix}
\frac{\partial h_1(X, U)}{\partial X_1} & \cdots & \frac{\partial h_1(X, U)}{\partial X_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial h_k(X, U)}{\partial X_1} & \cdots & \frac{\partial h_k(X, U)}{\partial X_n}
\end{bmatrix}_{(X_0, U_0)}
\]

\[
D = \begin{bmatrix}
\frac{\partial h_1(X, U)}{\partial U_1} & \cdots & \frac{\partial h_1(X, U)}{\partial U_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial h_k(X, U)}{\partial U_1} & \cdots & \frac{\partial h_k(X, U)}{\partial U_n}
\end{bmatrix}_{(X_0, U_0)}
\]

Where

Therefore, we have:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-g/L & 0 & 0 & 0 & LLg/L & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & g/L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/ML & 0 & 0 & 0 & 1/M \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/l_{joint} & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
The poles are the eigenvalues of $A$:

\[
\begin{align*}
0 & + 3.1321i \\
0 & - 3.1321i \\
3.1321 & \\
-3.1321 & \\
0 & \\
0 & 
\end{align*}
\]

Since there is a positive pole, the system is unstable. There are also imaginary poles indicating oscillations.

Using LQR with $Q = C^T C$ and $Q(1,1) = 10^5$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we get:

\[
K = \begin{bmatrix} 0 & 0 & 1569.6 & 501.1 & 0 & 0 \\ 314.8 & 9.6 & 0 & 0 & 63.1 & 20.9 \end{bmatrix}
\]

The poles for $(A - BK)$ are now:

\[
\begin{align*}
-0.1771 & + 3.2253i \\
-0.1771 & - 3.2253i \\
-1.6867 & + 1.5112i \\
-1.6867 & - 1.5112i \\
-3.1321 & + 0.0020i \\
-3.1321 & - 0.0020i 
\end{align*}
\]

Note that all the poles now have a negative real value component, indicating we are stable.

When simulating the system with disturbances of 10% the leg length and 10% of a full rotation (0.6283 radians) we get the following responses:
Action Parameterization 2

Given:

\[
\begin{align*}
M\ddot{x}_{\text{COM}} &= F_x \\
M\ddot{y}_{\text{COM}} &= F_y - Mg \\
I_{\text{COM}}\ddot{\theta} &= F_x y_{\text{COM}} - F_y x_{\text{COM}}
\end{align*}
\]

Choose our state to be:

\[
X = \begin{bmatrix} x_{\text{COM}} \\ \dot{x}_{\text{COM}} \\ y_{\text{COM}} \\ \dot{y}_{\text{COM}} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad U = \begin{bmatrix} X_1 \\ F_x \end{bmatrix}, \quad Y = \begin{bmatrix} x_{\text{COM}} \\ y_{\text{COM}} \end{bmatrix}
\]

State equations:

\[
\begin{align*}
\dot{X}_1 &= X_2 \\
\dot{X}_2 &= \frac{U_1}{M} \\
\dot{X}_3 &= X_4 \\
\dot{X}_4 &= \frac{U_2}{M} - g \\
\dot{X}_5 &= X_6 \\
\dot{X}_6 &= \frac{U_1 X_3 - U_2 X_1}{I_{\text{COM}}} - \frac{U_1}{I_{\text{COM}}}
\end{align*}
\]

Linearize about \((0, L + LL, 0)\) to get equilibrium \(X_e\) and \(U_e\), so:

\[
\begin{align*}
X_{1e} &= 0 \\
X_{3e} &= L + LL \\
X_{5e} &= 0
\end{align*}
\]
Setting $\dot{X} = 0$, we get:

\[
\begin{align*}
0 &= X_{2e} \\
0 &= U_{1e} \\
M g &= U_{2e} \\
0 &= X_{4e} \\
0 &= X_{6e}
\end{align*}
\]

Using the method of getting $A, B, C, D$ from parameterization 1, we have:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-M g/I_{coM} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1/M & 0 \\
0 & 0 \\
0 & 1/M \\
(L + LL)/I_{coM} & 0 \\
0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

The poles are the eigenvalues of $A$:

\[
\begin{align*}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{align*}
\]

All the poles are 0, which is undesirable since we don’t go back to the set point when we have a disturbance.

Using LQR with $Q = C^T C$ and $Q(1,1) = 10^5$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we get:

\[
K = \begin{bmatrix}
601.3509 & 354.3332 & 0 & 0 & -1 & -2.7681 \\
0 & 0 & 1 & 12.6491 & 0 & 0 \\
\end{bmatrix}
\]

The poles for $(A - BK)$ are now:

\[
\begin{align*}
-1.3963 + 1.4064 i \\
-1.3963 - 1.4064 i \\
-0.5138 + 0.4853 i
\end{align*}
\]
Note that all the poles now have a negative real value component, indicating we are stable.

When simulating the system with disturbances of 10% the leg length and 10% of a full rotation (0.6283 radians) we get the following responses:

**Action Parameterization 3**

This parameterization is the same as in action parameterization 1, except our control input is different. We are given:

\[
\tau_f = \frac{\tau}{2} + F_1 l_s \sin \left( \frac{\rho}{2} \right)
\]

\[
\tau_b = \frac{\tau}{2} - F_1 l_s \sin \left( \frac{\rho}{2} \right)
\]

So therefore:

\[
\tau_f + \tau_b = \tau
\]
$F_t = \frac{(\tau_f - \tau_b)}{2l_s \sin(\frac{\rho}{2})}$

$l_s$ is the four-bar segment length, which we assume to be 0.5m. At equilibrium, we assume $\rho = \pi/2$. Subbing these values into what we used in action parameterization 1, and using $U = [\tau_f]$, we have the state equations:

\[
\begin{align*}
\dot{X}_2 &= \frac{(U_1 - U_2)}{2l_s \sin(\frac{\rho}{2})} M \left( \frac{X_1 - LL \sin(X_5)}{L} \right) + \frac{U_1 + U_2}{ML} \left( \frac{X_3 - LL \cos(X_5)}{L} \right) \\
\dot{X}_4 &= \frac{(U_1 - U_2)}{2l_s \sin(\frac{\rho}{2})} M \left( \frac{X_3 - LL \cos(X_5)}{L} \right) - g + \frac{U_1 + U_2}{ML} \left( \frac{X_1 - LL \sin(X_5)}{L} \right) \\
\dot{X}_5 &= \frac{U_1 + U_2}{l_{joint}}
\end{align*}
\]

Linearize about $(0, L + LL, 0)$ to get equilibrium $X_e$ and $U_e$, so:

\[
\begin{align*}
X_{1e} &= 0 \\
X_{3e} &= L + LL \\
X_{5e} &= 0
\end{align*}
\]

Setting $\dot{X} = 0$, we get:

\[
\begin{align*}
0 &= X_{2e} \\
0 &= X_{4e} \\
0 &= X_{6e}
\end{align*}
\]

We also get:

\[
2l_s \sin(\frac{\rho}{2}) M g = U_{1e} - U_{2e}
\]

So therefore:

\[
\begin{align*}
l_s \sin(\frac{\rho}{2}) M g &= U_{1e} \\
-l_s \sin(\frac{\rho}{2}) M g &= U_{2e}
\end{align*}
\]

Using the method of getting $A, B, C, D$ from parameterization 1, we have:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-g/L & 0 & 0 & 0 & LLg/L & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & g/L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Since $A$ hasn’t changed, the poles remain the same as in part 1:

\[ 0 + 3.1321i \]
\[ 0 - 3.1321i \]
\[ 3.1321 \]
\[ -3.1321 \]
\[ 0 \]
\[ 0 \]

Since there is a positive pole, the system is unstable. There are also imaginary poles indicating oscillations.

Using LQR with $Q = C^T C$ and $Q(1,1) = 10^5$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we get:

\[
K = \begin{bmatrix} -204.0985 & 29.1640 & 554.9383 & 177.1780 & 42.7817 & 10.6257 \\ -204.0985 & 29.1640 & -554.9383 & -177.1780 & 42.7817 & 10.6257 \end{bmatrix}
\]

The poles for $(A - BK)$ are now:

\[-0.2085 + 3.2775i\]
\[-0.2085 - 3.2775i\]
\[-1.9881 + 1.7505i\]
\[-1.9881 - 1.7505i\]
\[-3.1321 + 0.0040i\]
\[-3.1321 - 0.0040i\]

Note that all the poles now have a negative real value component, indicating we are stable.

When simulating the system with disturbances of 10% the leg length and 10% of a full rotation (0.6283 radians) we get the following responses:
Part 1

Question 1:
I found that a $Q = C^T C$ and $Q(1,1) = 10^5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ worked best. Since only the relative values between Q and R matter, I only tried different values of Q. I changed only $Q(1,1)$ as I found that the X state did not go to zero fast enough, so I had to penalize the X state more with a higher Q-value for that state. Work best means converge to zero the fastest, preferably with few oscillations.

Question 2: I found that action parameterization 2 did poorly, and had much slower response times for disturbances in y. Disturbances in x caused large swings in theta, which was undesirable. However, for parameterization 2, the theta disturbances settled rather quickly with no oscillations.

Between parameterization 1 and 3, both are very similar but on closer inspection, parameterization 3 is slightly better. Below are the graphs of the x-disturbance and the theta-disturbance plotted side by side with parameterization 1 on the left and parameterization 3 on the right:
It seems that parameterization 3 has slightly larger amplitudes in the X state after the disturbance, indicating slower convergence to zero. Again, work best means converge to zero the fastest, preferably with few oscillations.

Question 3:
It seems that the x value converges the slowest, and therefore it is most vulnerable in this state. Here, vulnerable means that it takes longer to stabilize disturbances in this direction.

Part 2
I have selected an integration time step of 0.01s. I also found that I had to change the Q value for the LQR to \( Q(1,1) = 10^7 \) to get good results because, as shown in part 1, the Q values converged over the course of 10s, which was too long.

Below is a plot of the cost calculated as the sum of the squared magnitudes of the U vectors for various step sizes for parameterization 1:
The smaller the step size, the smaller the cost. However, a small step size isn’t really realistic for walking, as it is more akin to shuffling. Therefore, I limited the results to only step sizes between 0.9 and 1.1m. The optimal step size found was at 0.905m. Below is a plot of walking done by parameterization 1 along with the cost from 0.9m to 1.1m.

The step size was optimized for parameterization 2 and 3 in the same way for their walking plots:
Part 3

We are tracking the desired position that is the path computed from Part 2.
Compared to before, the trajectory is smoother with less backtracking than before. The cost also decreased. The cost is 20.17%, 28.96%, and 11.67% the previous cost for parameterizations 1, 2 and 3 respectively. It seems that parameterization 3 had the best cost reduction, and performs the best out of the three with lower steady state error at 10m and lower overall cost. Parameterization 3 is 64% of the cost of parameterization 1, and 24.69% of the cost of parameterization 2. However, comparing the cost is less meaningful because we are comparing torques with leg force as if they are equivalent. It might be harder to produce torques than leg force, or vice versa, in the physical system.

The figures below show the trajectories for parameterization 1 after one forward, one backwards, and another forward pass with DDP. The trajectory seems smoother, but DDP should also predict the future better. I am unsure why it moves backwards a bit at 1s, it should be ahead of the LQR controller instead of slightly behind it. It might be because of how my stepping was defined.