USING THE SERRE SPECTRAL SEQUENCE FOR COMPUTING (CO)HOMOLOGY

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ABSTRACT. Here I present a very brief overview of the Serre spectral sequence and give an example of how we can use it to compute the (co)homology the complex projective space, and of the loop space of a sphere along with an alternative technique that uses the fact that $\mathbb{F}P^n \cong \Omega S^n$. I assume a pretty basic understanding of algebraic topology at the level of Hatcher's Algebraic Topology. As always, please let me know if you find any issues!

1. THE SERRE SPECTRAL SEQUENCE

When we are given an arbitrary topological space it can often be very difficult to compute the homology or cohomology of the space without the appropriate tools. One of the most important tools for computing (co)homology are spectral sequences.

Definition 1.1. A spectral sequence over a fixed abelian category (such as modules or rings) is a collection of pages $\{E_1, \ldots, E_n\}$ (in this category) for $n \geq 2$ equip with the following properties and objects:

1. $E_r^{s,t} = 0$ if $s < 0$ or $t < 0$.
2. a differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$ with $d_r^2 = 0$.
3. $E_r^{s,t}$ is the homology of $d_r$. That is:
   $$E_r^{s,t} \cong \text{ker}(d_r) / \text{im}(d_r) = E_r^{s-r,t+r-1} \to E_r^{s,t}.$$
4. a product $E_r^{s,\nu} \otimes E_r^{u,\nu} \to E_r^{s+u,\nu}$ making $E_r$ a bigraded ring that is an anti-commutative ring, i.e.
   $$ab = (-1)^{(s+\nu)(a+\nu)}ba$$

where $a \in E_r^{s,\nu}$ and $b \in E_r^{u,\nu}$ such that the differential is a derivation, i.e.
$$d_r(ab) = d_r(a)b + (-1)^{s+\nu}ad_r(b).$$

We say that a spectral sequence converges to a graded abelian group $A$, if there’s a filtration $A = F^0A^k \supseteq P^1A^k \supseteq \cdots \supseteq P^kA^k \supseteq F^{k+1}A^k = 0$, such that $F^kA^k \cdot F^mA^m \leq F^{k+m}A^{k+m}$ and there are isomorphisms $F^kA^k / F^{k+1}A^k \cong E^\infty_{r=k}$. The Serre spectral sequence is one of the first spectral sequences that one encounters (it certainly was the first one that I learned about!). The Serre spectral sequence relates the homology (or cohomology) of $F$, $E$, and $B$ of a fibration $F \to E \to B$ under some additional assumptions. The entries of the Serre spectral sequence’s second page are $E_2^{s,t} = H^s(B; H^t(F; G))$

Theorem 1.2. Let $F \to E \to B$ be a fibration with $B$ simply connected and $\pi_1(B)$ acting trivially on $H^*(F; R)$. Then there exists a spectral sequence $E_2^{s,t} = H^s(B; H^t(F; R)) \implies H^{s+t}(E; R)$.

Although the formula $H^s(B; H^t(F; R))$ looks strange it is actually completely natural. Recall the universal coefficient theorem for cohomology: that is there is an exact sequence
$$0 \to \text{Ext}(H_{n-1}(X), G) \to H^n(X; G) \to \text{Hom}(H_n(X), G) \to 0.$$ 
So we have that if the Ext-groups and the Tor-groups vanish and the cohomology groups are finitely generated then
$$H^s(B; H^t(F; R)) \cong H^s(B; R) \otimes H^t(F; R).$$

This is true in particular if $R$ is a field.

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The theorem above is the Serre spectral sequence for cohomology. One can similarly define a Serre spectral sequence for homology completely analogously:

$$E^2_{s,t} = H_s(B; H_t(F; R)) \Longrightarrow H_{s+t}(E; R).$$

The notation for the Serre homological and cohomological sequences are dual (upper indices vs. lower indices). Again, using the universal coefficient theorem for homology we have a natural short exact sequence

$$0 \to H_1(B; \mathbb{Z}) \otimes H_t(F; \mathbb{Z}) \to H_1(B; H_t(F, \mathbb{Z})) \to \text{Tor}^\mathbb{Z}(H_{s-1}(B), H_t(F)) \to 0.$$  

So we have that $H_1(F; Z)$ or $H_{s-1}(B; Z)$ is torsion free then $E^2_{s,t} \cong H_s(B; Z) \otimes H_t(F; Z) \Longrightarrow H_{s+t}(E; Z)$.

Note that this spectral sequence has a product in the following sense: suppose that $E^r_{s,t} \otimes (E^r_{u,v})_2 \to (E^r_{s+u,t+v})_3.$

Consider the fibration

$$\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n.$$  

The Serre spectral sequence then reads that

$$H^i(\mathbb{C}P^n; H^i(\mathbb{S}^1)) \Longrightarrow H^{i+1}(\mathbb{S}^{2n+1}),$$

where we are implicitly assuming coefficients in $\mathbb{Z}$. Now the group $H^i(\mathbb{C}P^n; H^i(\mathbb{S}^1))$ is $H^i(\mathbb{C}P^n)$ if $t=0,1$ and is zero otherwise.

So we have that the $E_2$ page looks like

\[
\begin{matrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{matrix}
\]

\[
\begin{matrix}
0 & H^0\mathbb{C}P^n & H^1\mathbb{C}P^n & H^2\mathbb{C}P^n & \cdots & H^{2n-2}\mathbb{C}P^n & H^{2n-1}\mathbb{C}P^n & H^{2n}\mathbb{C}P^n & 0 \\
0 & H^0\mathbb{C}P^n & H^1\mathbb{C}P^n & H^2\mathbb{C}P^n & \cdots & H^{2n-2}\mathbb{C}P^n & H^{2n-1}\mathbb{C}P^n & H^{2n}\mathbb{C}P^n & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{matrix}
\]

Where the only nontrivial differentials are shown as above (the same pattern). Since $H^k(\mathbb{S}^{2n+1}) \cong \mathbb{Z}$ if $k = 0, 2n+1$ and is zero otherwise, and since $H^1(\mathbb{C}P^n)$ lies on the line $s + t = 1$, we see that $H^1(\mathbb{C}P^n) \cong 0$. From this it follows that $H^{2k+1}\mathbb{C}P^n \cong 0$ for all $k$. Clearly via $d_2$ each even cohomology group of $\mathbb{C}P^n$ must be isomorphic so it suffices to determine $H^0\mathbb{C}P^n$. But this is simply $\mathbb{Z}$, which finishes our computations of all of the cohomology groups of $\mathbb{C}P^n$.

2. Computing $H^*(\mathbb{C}P^n; \mathbb{Z})$.

3. (Co)homology of $\Omega S^n$
Proof. Consider the path-loop fibration $\Omega S^n \to PS^n \to S^n$, and note that the path space $PS^n$ is contractible. Hence the Serre spectral sequence says that $E^2_{s,t} = H_*(S^n; H_*(\Omega S^n)) \Rightarrow H_*(\Omega S^n)$. We know that $E^2_{s,t} = H_*(\Omega S^n)$ if $p = 0, n$, and is zero otherwise. Consider the generator $\sigma$ of $H_0(S^n)$; Then $H_0(\Omega S^n) \cong (\sigma)$. Note that almost all of the differentials are zero, except $d^n : E^n_{0,t} \to E^n_{0,t+n-1}$. We therefore have the following page

\[
\begin{array}{ccc}
H_{k+n-1}(\Omega S^n) & 0 & H_{k+n-1}(\Omega S^n) \\
\vdots & \vdots & \vdots \\
H_{n-1} \Omega S^n & H_{n-1} \Omega S^n \\
\vdots & \vdots & \vdots \\
H_k(\Omega S^n) & H_k(\Omega S^n) \\
\vdots & \vdots & \vdots \\
H_0(\Omega S^n) & 0 & H_0(\Omega S^n)
\end{array}
\]

We claim that all the nonvanishing maps are isomorphisms. To see this note that

$E^{n+1}_{s,t} \cong \cdots \cong E^\infty_{s,t} = E^0 H_{s,t} (PS^n)/E_{s-1} H_{s,t} (PS^n)$.

But since $PS^n$ is contractible, we have that $E^\infty_{s,t} \cong 0$ unless $s = t = 0$. Therefore if $d^n$ was not an isomorphism, we would get nontrivial elements in $E^{n+1}_{s,t}$. Therefore, $H_k(\Omega S^n) \cong \mathbb{Z}$ if $k$ is a multiple of $(n-1)$, and is zero otherwise. Say that $x_1$ generates $E^2_{0,t(n-1)}$. Then the generator of $E^2_{n,t(n-1)}$ is $\sigma \otimes x_1$, and thus the differential goes $d(\sigma \otimes x_1) = x_{t+1}$.

Now we will look at the multiplicative structure on $H_*(\Omega S^n)$. We have to choose three Hopf fibrations; indeed, we can pick the most obvious ones: $\Omega S^n \to PS^n \to S^n$, $\Omega S^n \to PS^n \to S^n$ and $\Omega S^n \to \Omega S^n \to \Omega S^n$. It is easy to see that the desired diagram commutes so that we have a product on spectral sequences. For the fibration $\Omega S^n \to \Omega S^n \to \Omega S^n$, it is obvious that $E^k_{0,t} = H_t(\Omega S^n)$ for all $t$ and $E^k_{p,t} = 0$ for all $p > 0$, and all differentials are zero.

Now if $E^k$ is the Serre spectral sequence for $\Omega S^n \to PS^n \to S^n$, the multiplication is $H_n(\Omega S^n) \otimes E^r_{u,v} \to E^r_{u+v,v}$. Since $x_1 \otimes \sigma = \sigma \otimes x_1 \in E^2_{n,t(n-1)}$ (under the universal coefficient theorem mapping)

$E^2_{n,0} \otimes E^2_{0,t(n-1)} = H_n(S^n) \otimes H_{t(n-1)}(\Omega S^n) \cong H_n(S^n; H_{t(n-1)}(\Omega S^n)) = E^2_{n,t(n-1)}$,

and since we know that $d(\sigma \otimes x_1) = x_{t+1}$, it follows that

$x_{t+1} = dx_1 \otimes \sigma \pm x_1 \otimes d\sigma = \pm x_1 \otimes x_1$

because $dx_1 = 0$ and $d\sigma = x_1$. So by induction, we have that $x_t = \pm x_1^t$. The desired result follows immediately. \qed

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Serre spectral sequence

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