1. Introduction

Our goal is to define exponentials of linear operators. We will try to construct \( e^{tA} \) as a linear operator, where \( A : \mathcal{D}(A) \to X \) is a general linear operator, not necessarily bounded. Notationally, it seems like we are looking for a solution to \( \dot{\mu}(t) = A\mu(t), \mu(0) = \mu_0 \), and we would like to write \( \mu(t) = e^{tA}\mu_0 \). It turns out that this will hold once we make sense of the terms.

How can we construct \( e^{tA} \) when \( A \) is a finite matrix? The most obvious way is to write down the power series:

\[
\sum_{n=0}^{\infty} \frac{1}{n!} (tA)^n.
\]

This series is absolutely convergent for every \( A \) and \( t \in \mathbb{R} \). In fact, this method works for \( A \in \mathcal{L}(X;X) \), even if \( X \) is infinite dimensional.

A second method is to consider the connection with the explicit Euler scheme. Consider the system of ordinary differential equations:

\[
\begin{cases}
\dot{\mu}(t) = A\mu(t), \\
\mu(0) = \mu_0.
\end{cases}
\]

Partition \([0,t]\) into \( n \) parts and write

\[
\dot{\mu}\left(\frac{kt}{n}\right) = \frac{n}{t} \left( \mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right) \right),
\]

the forward difference quotient approximation. From the ODE, we get

\[
A\mu\left(\frac{kt}{n}\right) = \frac{n}{t} \left( \mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right) \right),
\]

\[
\mu\left(\frac{(k+1)t}{n}\right) = \left(1 + \frac{t}{n}A\right)\mu\left(\frac{kt}{n}\right),
\]

\[
\mu(t) = \mu\left(\frac{nt}{n}\right) \approx \left(1 + \frac{t}{n}A\right)^n \mu_0.
\]

Thus \( \mu(t) = \lim_{n \to \infty} \left(1 + \frac{t}{n}A\right)^n \mu_0 \) and we write \( e^{tA} = \lim_{n \to \infty} \left(1 + \frac{t}{n}A\right)^n \).

Both of these methods are doomed to fail if \( A \) is not bounded. When the explicit method fails, one would normally try the implicit method. The third method we consider is the connection with the implicit Euler scheme. Partition \([0,t]\) into \( n \) parts and write

\[
\dot{\mu}\left(\frac{(k+1)t}{n}\right) = \frac{n}{t} \left( \mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right) \right),
\]

the backward difference quotient approximation. From the ODE, we get

\[
A\mu\left(\frac{(k+1)t}{n}\right) = \frac{n}{t} \left( \mu\left(\frac{(k+1)t}{n}\right) - \mu\left(\frac{kt}{n}\right) \right),
\]

\[
\mu\left(\frac{(k+1)t}{n}\right) = \left(1 - \frac{t}{n}A\right)^{-1} \mu\left(\frac{kt}{n}\right),
\]

\[
\mu(t) = \mu\left(\frac{nt}{n}\right) \approx \left(1 - \frac{t}{n}A\right)^{-n} \mu_0.
\]

Thus \( \mu(t) = \lim_{n \to \infty} \left(1 - \frac{t}{n}A\right)^{-n} \mu_0 \) and we write \( e^{tA} = \lim_{n \to \infty} \left(1 - \frac{t}{n}A\right)^{-n} \). This works for some unbounded \( A \) as well. The key point will be the behavior of \( \|R(\lambda;A)^n\| \) for large \( n \).

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An engineer might consider the Laplace transform. If \( f(t) = e^{tA} \) then it can be shown that \( \hat{f}(\lambda) = (\lambda I - A)^{-1} = R(\lambda; A) \). There is an inversion formula, namely

\[
e^{tA} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R(\lambda; A) \, d\lambda,
\]

where \( \gamma \) is chosen such that the spectrum of \( A \) lies to the left of the line over which we are integrating. This formula can be interpreted and works for many important unbounded operators.

A fifth method works for self-adjoint matrices. Let \( \{e_k\}_{k=1}^N \) be an orthonormal basis of \( X \) of eigenvectors of \( A \). For any \( v \in X, v = \sum_{k=1}^N (v, e_k)e_k \) and \( Av = \sum_{k=1}^N \lambda_k (v, e_k)e_k \). We take

\[
e^{tA}v = \sum_{k=1}^N e^{\lambda_k t}(v, e_k)e_k.
\]

In general, if \( X \) is a Hilbert space and \( A : \mathcal{D}(A) \to X \) is self-adjoint then

\[
A = \int_{-\infty}^{\infty} \lambda \, dP(\lambda),
\]

where \( \{P(\lambda) : \lambda \in \mathbb{R} \} \) is the spectral family associated with \( A \). We know that \( \sigma(A) \subseteq \mathbb{R} \), so if \( \sigma(A) \) is bounded above then we could define

\[
e^{tA} = \int_{-\infty}^{\infty} e^{\lambda t} \, dP(\lambda).
\]

Note that the matrix \( A \) can be recovered from its exponential via the formula

\[
A = \lim_{t \to 0} \frac{1}{t} \left( e^{tA} - 1 \right).
\]

2. LINEAR \( C_0 \)-SEMIGROUPS

Let \( X \) be a Banach space over \( K \), where \( K = \mathbb{R} \) or \( K = \mathbb{C} \).

**Definition 2.1.** A linear \( C_0 \)-semigroup (or a strongly continuous semigroup) is a mapping \( T : [0, \infty) \to \mathcal{L}(X; X) \) such that

(i) \( T(0) = 1 \),

(ii) \( T(t + s) = T(t)T(s) \) for all \( s, t \in [0, \infty) \), and

(iii) for all \( x \in X \), \( \lim_{t \to 0} T(t)x = x \).

\[\Box\]

**Remark 2.2.**

(i) By the second condition \( T(t)T(s) = T(s)T(t) \) for all \( s, t \).

(ii) Sometimes we will use the notation \( \{T(t)\}_{t \geq 0} \).

(iii) If we have a mapping \( T : [0, \infty) \to \mathcal{L}(X; X) \) satisfying conditions (i) and (ii), (called a semigroup of bounded linear operators) then if the following condition holds so does (iii).

(iii') \( \lim_{t \to 0} \langle x^*, T(t)x \rangle = \langle x^*, x \rangle \) for all \( x^* \in X^* \) and \( x \in X \).

(iv) The condition \( \lim_{t \to 0} ||T(t) - I|| = 0 \) implies that \( T(t) = \sum_{n=0}^{\infty} \frac{1}{n!} (tA)^n \) for all \( t \), for some \( A \in \mathcal{L}(X; X) \). This condition is too strong for practical purposes.

(v) The “\( C_0 \)” in the name may come form “continuous at zero” or it may refer to the fact that these semigroups are (merely) continuous, as opposed to differentiable, etc.

\[\Box\]

Let \( T \) be a linear \( C_0 \)-semigroup. The infintesimal generator of \( T \) is the linear operator \( A : \mathcal{D}(A) \to X \) defined as follows.

\[
\mathcal{D}(A) := \left\{ x \in X : \lim_{t \to 0} \frac{1}{t} (T(t)x - x) \right\}
\]
and for all $x \in \mathcal{D}(A), Ax = \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - x)$. It is not immediately obvious that $\mathcal{D}(A) \neq \{0\}$. We will show that $\mathcal{D}(A)$ is dense and that $A$ is a closed linear operator.

**Example 2.3.** Let $X = BUC(\mathbb{R})$ =bounded uniformly continuous functions $\mathbb{R} \rightarrow \mathbb{K}$. Define $(T(t)f)(x) := f(t + x)$ for all $t \in [0, \infty)$ and $x \in \mathbb{R}$. Clearly $T$ satisfies (i) and (ii) of the definition. Uniform continuity is essential to get (iii). Indeed, if $f$ is uniformly continuous then

$$\|T(t)f - f\|_\infty = \sup \{|f(t + x) - f(x)| : x \in \mathbb{R}\} \rightarrow 0 \text{ as } t \rightarrow 0.$$ 

The infinitesimal generator is

$$Af = \lim_{t \downarrow 0} \frac{f(t + x) - f(x)}{t} = f'(x),$$

i.e. differentiation. Note that the solution to the PDE $\mu_t(x, t) = \mu_x(x, t), \mu(x, 0) = \mu_0$ is $\mu(x, t) = \mu_0(x + t) = (T(t)\mu_0)(x)$.

**Lemma 2.4.** Let $T$ be a linear $C_0$-semigroup. Then there are $M, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \in [0, \infty)$.

**Proof.** We claim that there is some $\eta > 0$ such that $\sup \{|T(t)| : t \in [0, \eta]\}$ is finite. Indeed, assume for the sake of contradiction there is no such $\eta$. Choose $(t_n)_{n=1}^{\infty}$ such that $t_n \downarrow 0$ and $(T(t_n)x)_{n=1}^{\infty}$ is unbounded. However, for all $x \in X$, since $T(t_n)x \rightarrow x$, $(T(t_n)x)_{n=1}^{\infty}$ is a convergent sequence, so $\sup \{|T(t_n)x| : n \in \mathbb{N}\}$ is finite for each $x \in X$. By the Banach-Steinhaus theorem we deduce that $\sup \{|T(t_n)||t_n| : n \in \mathbb{N}\}$ is finite, a contradiction.

Now let $\eta > 0$ be as above. Set $M := \sup \{|T(t)| : t \in [0, \eta]\} \geq 1$. Let $t \in (0, \infty)$ be given. Choose $n \geq 0$ and $\alpha \in [0, \eta]$ such that $t = n\eta + \alpha$. Then $T(t) = T(n\eta + \alpha) = (T(n\eta))T(\alpha)$ by the semigroup property. Hence,

$$\|T(t)\| \leq \|T(n\eta)\|\|T(\alpha)\| \leq MM^n.$$ 

Now let $\omega = \frac{1}{\alpha} \log M \geq 0$, so that $\omega t \geq n \log M$, and $\|T(t)\|Me^{\omega t}$. \hfill $\diamond$

**Definition 2.5.** Let $T$ be a linear $C_0$-semigroup. We say that $T$ is

(i) uniformly bounded if there is $M \in \mathbb{R}$ such that $\|T(t)\| \leq eM$ for all $t \geq 0$.

(ii) contractive if $\|T(t)\| \leq 1$ for all $t \geq 0$.

(iii) quasi-contractive provided there is $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

\hfill $\diamond$

Contractive semigroups are much easier to study than general linear $C_0$-semigroups. If $T$ is a linear $C_0$-semigroup satisfying $\|T(t)||eM e^{\omega t}$ then $S(t) := e^{-\omega t}T(t)$ is a uniformly bounded linear $C_0$-semigroup. Note that the infinitesimal generator of $S$ is related to that of $T$ as follows.

$$\lim_{t \downarrow 0} \frac{S(t)x - x}{t} = \lim_{t \downarrow 0} \frac{e^{-\omega t}T(t)x - x}{t} = \lim_{t \downarrow 0} \frac{e^{-\omega t} - 1}{t} T(t)x + \lim_{t \downarrow 0} \frac{T(t)x - x}{t} = -\omega x + Ax = (A - \omega I)x.$$ 

Further, there is an equivalent norm $\| \cdot \|$ on $X$ such that $S$ is contractive with respect to $\| \cdot \|$. In fact, we may take $\|x\| := \sup \{|S(t)x| : t \in [0, \infty]\}$. Indeed, for all $x \in X$,

$$\|S(t)x\| = \sup \{|S(t + s)x| : s \in [0, \infty]\} \leq \|x\|.$$ 

**Warning:** The norm $\| \cdot \|$ need not preserve all “nice” geometric properties of $\| \cdot \|$, such as the parallelogram law.

**Lemma 2.6.** Let $T$ be a linear $C_0$-semigroup and let $x \in X$ be given. Then the mapping $t \mapsto T(t)x$ is continuous on $[0, \infty)$.

**Proof.** For continuity from the right, let $t \geq 0$ be given and notice that

$$T(t + h)x - T(t)x = (T(h) - I)(T(t)x) \rightarrow 0 \text{ as } h \rightarrow 0.$$ 

\hfill 3
Lemma 2.7. Let $T$ be a linear $C_{0}$-semigroup with infinitesimal generator $A$, and let $x \in X$ be given. For all $t \geq 0$, $\frac{1}{h} \int_{t}^{t+h} T(s)x \, ds = T(t)x$ (where the limit is one sided if $t = 0$).

Proof. Let $t > 0$ be given. For $h > 0$,
\[
\frac{T(h)-1}{h} \int_{0}^{t} T(s)x \, ds = \frac{1}{h} \int_{0}^{t} (T(s+h) - T(s))x \, ds
\]
\[
= \frac{1}{h} \int_{0}^{t} T(s+h)x \, ds - \frac{1}{h} \int_{0}^{t} T(s)x \, ds
\]
\[
= \frac{1}{h} \int_{h}^{t} T(u)x \, du + \frac{1}{h} \int_{t}^{t+h} T(u)x \, du - \frac{1}{h} \int_{0}^{t} T(s)x \, ds - \frac{1}{h} \int_{0}^{h} T(s)x \, ds
\]
\[
= \frac{1}{h} \int_{t}^{t+h} T(u)x \, du - \frac{1}{h} \int_{0}^{h} T(s)x \, ds
\]
\[
to T(t)x - x as h \to 0
\]
by part (a). The conclusion immediately follows.

Lemma 2.8. Let $T$ be a linear $C_{0}$-semigroup with infinitesimal generator $A$, and $x \in \mathcal{D}(A)$ be given. Put $\mu(t) = T(t)x$ for all $t \geq 0$. Then $\mu(t) \in \mathcal{D}(A)$ for all $t \geq 0$, $\mu$ is differentiable on $[0, \infty)$, and for each $t \geq 0$,
\[
\dot{\mu}(t) = T(t)Ax = AT(t)x = A\mu(t).
\]

Proof. Let $t \geq 0$ be given. For $h > 0$,
\[
\frac{T(t+h)x - T(t)x}{h} = \left(\frac{T(h)-1}{h}\right)T(t)x = T(t)\left(\frac{T(h)-1}{h}\right)x \to T(t)Ax
\]
as $h \downarrow 0$. In particular, $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$. Furthermore, $D^+\mu(t) = x = T(t)Ax$. Let $t > 0$ be given. For $h \in (0, t)$,
\[
\frac{T(t-h)x - T(t)x}{h} = T(t-h)\left(\frac{x-T(h)x}{h}\right) \to -T(t)Ax as h \to 0.
\]
So we deduce that $D^-\mu(t)x = T(t)Ax$. Since the left and right derivatives both exist and are equal, $\mu$ is differentiable and $\dot{\mu}(t) = A\mu(t)$.

Lemma 2.9. Let $T$ be a linear $C_{0}$-semigroup with infinitesimal generator $A$, and let $x \in \mathcal{D}(A)$ be given. Then for all $s, t \in [0, \infty)$,
\[
T(t)x - T(s)x = \int_{s}^{t} AT(u)x \, du = \int_{s}^{t} T(u)Ax \, du.
\]

Proof. This follows from Lemma 2.8 and the fundamental theorem of calculus.
Theorem 2.10. Let $T$ be a linear $C_0$-semigroup with infinitesimal generator $A$. Then $\mathcal{D}(A)$ is dense in $X$ and $A$ is closed.

Proof. Let $x \in X$. By Lemma 2.7 we see that $x = \lim_{h \to 0} \int_0^h T(s)x \, ds$, and $\int_0^h T(s)x \, ds \in \mathcal{D}(A)$ for all $h \geq 0$, so $\mathcal{D}(A)$ is dense in $X$.

Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{D}(A)$ converging to $x \in X$ and suppose that $Ax_n \to y \in X$ as $n \to \infty$. We must show that $x \in \mathcal{D}(A)$ and that $Ax = y$. For $h > 0$, by Lemma 2.9,

$$T(h)x_n - x_n = \int_0^h T(s)Ax_n \, ds,$$

so by Lemma 2.7,

$$Ax = \lim_{h \to 0} \frac{T(h)x - x}{h} = \lim_{h \to 0} \frac{1}{h} \int_0^h T(s)y \, ds = y.$$

It follows that $x \in \mathcal{D}(A)$ and $Ax = y$. \hfill $\square$

Lemma 2.11. Let $S$, $T$ be linear $C_0$-semigroups having the same infinitesimal generator $A$. Then $S(t) = T(t)$ for all $t \geq 0$.

Proof. Let $x \in \mathcal{D}(A)$ and $t > 0$ be given. Define the function $\mu : [0, t] \to X$ by $\mu(s) = T(t-s)S(s)x$ for all $x \in [0, t]$. We will show that $\mu$ is constant as follows. We claim that $\mu$ is differentiable on $[0, t]$ and

$$\frac{d}{ds} \mu(s) = T(t-s)AS(s)x - T(t-s)AS(s)x = 0$$

for all $s \in [0, t]$. This will imply that $\mu$ is constant on $[0, t]$, so

$$T(t)x = \mu(0) = \mu(1) = S(t)x.$$

Since $\mathcal{D}(A)$ is dense in $X$, it will follow that $T(t) = S(t)$ on $X$ for all $t \geq 0$. To prove the claim we apply Lemma 2.8.

$$\frac{\mu(s+h) - \mu(s)}{h} = \frac{1}{h} (T(t-s-h)S(s+h)x - T(t-s)S(s)x)
= \frac{1}{h} T(t-s-h)(S(s+h)-S(s))x + \frac{1}{h} (T(t-s-h) - T(t-s))S(s)x
= T(t-s-h) - S(s)x + T(t-s-h)S(s)x
= T(t-s)AS(s)x - T(t-s)AS(s)x = 0 \text{ as } h \to 0.$$

The mean value theorem holds for calculus in Banach spaces, and so $\mu$ is constant. \hfill $\square$

3. Infinitesimal Generators

Given a closed densely defined $A$, how do we tell if $A$ generates a linear $C_0$-semigroup? Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$ and put $f(t) = t^{n-1}e^{at}$ for all $t \geq 0$. Recall that the Laplace transform of $f$ is

$$\hat{f}(\lambda) = \frac{(n-1)!}{(\lambda-a)^n}.$$

Let $A$ be an $N \times N$ matrix and put $F(t) = e^{tA}$.

$$\hat{F}(\lambda) = \int_0^\infty e^{-\lambda t} e^{tA} \, dt = \int_0^\infty e^{t(A-\lambda 1)} \, dt = (A - \lambda 1)^{-1} e^{t(A-\lambda 1)} \bigg|_0^\infty = -(A - \lambda 1)^{-1} = R(\lambda; A).$$

Recall that $e^{tA} = \lim_{n \to \infty} \left( I - \frac{t}{n} \right)^{-n} = \lim_{n \to \infty} \left( \frac{t}{n} \right)^n R\left( \frac{t}{n}, A \right)^n$. To apply this to unbounded operators, the behavior of $R(\lambda; A)^n$ for large $n$ will be key. We conjecture that

$$R(\lambda; A)^n = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} e^{tA} \, dt.$$
Lemma 3.1. Let $M, \omega \in \mathbb{R}$ and $\lambda \in \mathbb{K}$ with $\Re(\lambda) > \omega$ be given. Let $T$ be a linear $C_0$-semigroup such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and let $A$ be the infinitesimal generator of $T$. Then $\lambda \in \rho(A)$ and, for all $x \in X$,

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$ 

Proof. Put $I_1(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$ for all $x \in X$. We need to show that $\lambda \in \rho(A)$ and $R(\lambda; A) = I_1(\lambda)$. Let $x \in \mathcal{D}(A)$ be given.

$$I_1(\lambda)Ax = \int_0^\infty e^{-\lambda t} T(t)Ax \, dt = \int_0^\infty e^{-\lambda t} \frac{d}{dt}(T(t)x) \, dt = -x + \lambda \int_0^\infty e^{-\lambda t} T(t)x \, dt = \lambda I_1(\lambda)x - x.$$ 

Now let $x \in X$ be given. We will show that $I_1(\lambda)x \mathcal{D}(A)$ and

$$AI_1(\lambda)x = \lambda I_1(\lambda)x - x.$$ 

Fix $h > 0$ and compute the difference quotient:

$$\left( \frac{T(h) - I_1(\lambda)}{h} \right) I_1(\lambda)x \xrightarrow[h \to 0]{} \frac{1}{h} \int_0^\infty e^{-\lambda t} (T(t + h)x - T(t)x) \, dt = \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t + h)x \, dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt = \frac{1}{h} \int_0^\infty e^{-\lambda(s-h)} T(s)x \, ds - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt = \frac{1}{h} \int_0^\infty e^{-\lambda(s-h)} T(s)x \, ds - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt = \frac{1}{h} \int_0^\infty e^{-\lambda(t-h)} T(t)x dt - \frac{1}{h} \int_0^h e^{-\lambda t} T(t)x dt = \int_0^\infty e^{-\lambda(t-h)} - e^{-\lambda t} \frac{h}{h} T(t)x dt \to \lambda I_1(\lambda)x - x as h \to 0.$$ 

This proves the lemma. \qed

Lemma 3.2. Let $M, \omega \in \mathbb{R}$ and $\lambda \in \mathbb{K}$ with $\Re(\lambda) > \omega$ be given. Let $T$ be a linear $C_0$-semigroup such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and let $A$ be the infinitesimal generator of $T$. Then $\lambda \in \rho(A)$ and, for all $n \in \mathbb{N}$ and all $x \in X$,

$$R(\lambda; A)^n x = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x \, dt.$$ 

Proof. We already know that $\rho(A) \ni \left\{ \mu \in \mathbb{K} : \Re(\mu) > \omega \right\}$. We also know that $\mu \to R(\mu; A)$ is analytic. In particular, we have

$$R(\mu; A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda; A)^{n+1} = \sum_{n=0}^{\infty} R(\lambda; A)^{n+1} (\mu - \lambda)^n$$

for $|\mu - \lambda|$ sufficiently small. Let $R^{(k)}(\lambda; A)$ denote the $k$th derivative of $R(\mu; A)$ evaluated at $\mu = \lambda$. From the power series, for all $n \in \mathbb{N}$,

$$\frac{R^{(n-1)}(\lambda; A)}{(n-1)!} = (-1)^{n-1} R(\lambda; A)^n.$$
By Lemma 3.1, \( R(\lambda;A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \) for all \( x \in X \). From this,

\[
R^{(n-1)}(\lambda;A)x = (-1)^{n-1} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x \, dt.
\]

This proves the result. \(\square\)

**Theorem 3.3** (Hille-Yosida, 1948). Let \( M, \omega \in \mathbb{R} \) be given. Suppose that \( A : \mathcal{D}(A) \to X \) is a linear operator with \( \mathcal{D}(A) \subseteq X \). Then \( A \) is the infinitesimal generator of a linear \( C_0 \)-semigroup \( T \) satisfying \( \|T(t)\| \leq M e^{\omega t} \) for all \( t \geq 0 \) if and only if the following hold.

(i) \( A \) is closed and \( \mathcal{D}(A) \) is dense in \( X \); and

(ii) \( \rho(A) \supseteq \{ \lambda \in \mathbb{R} : \lambda > \omega \} \) and \( \|R(\lambda;A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \) for all \( \lambda \in \mathbb{R} \) with \( \lambda > \omega \) and all \( n \in \mathbb{N} \).

**Remark 3.4.** Note that the condition that \( \|R(\lambda;A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \) may be difficult to verify in practice. Notice that \( \|R(\lambda;A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \) implies that \( \|R(\lambda;A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \), so if \( M = 1 \), i.e. if the semigroup is quasi-contractive, then it is enough to verify the inequality for \( n = 1 \) only. \(\diamond\)

**Proof.**

**Step 1. Necessity.**

We have already seen that (i) holds, by Theorem 2.10, and that \( \rho(A) \) contains \( \{ \lambda \in \mathbb{R} : \lambda > \omega \} \), by Lemma 3.1. By Lemma 3.2,

\[
R(\lambda;A)^n = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x \, dt,
\]

\[
\|R(\lambda;A)^n\| \leq \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} \|T(t)x\| \, dt
\]

\[
\leq \frac{M}{(n-1)!} \|x\| \int_0^\infty e^{-\lambda t} t^{n-1} e^{\omega t} \, dt
\]

\[
= \frac{M}{(n-1)!} \|x\| \int_0^\infty e^{-\lambda t} t^{n-1} \, dt
\]

\[
= \frac{M}{(n-1)!} \|x\| \frac{(n-1)!}{(\lambda-\omega)^n}
\]

This concludes the proof of necessity.

**Step 2. Sufficiency.**

Should we try using the inverse Laplace transform? If we could write

\[
T(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} R(\lambda;A) \, d\lambda
\]

then \( T \) would have higher order regularity in general. This method would work for so called “analytic” semigroups, but not for general \( C_0 \)-semigroups.

How about the limit obtained from considering the implicit scheme? In general \( T(t) = \lim_{n \to \infty} \left( I - \frac{t}{\lambda} A \right)^n \), and this method can be used, but we will not use it here. What we will do is approximate \( A \) with bounded operators \( \{A_\lambda\}_{\lambda > \omega} \) and put \( T_\lambda(t) = \sum_{n=0}^\infty \frac{t^n}{n!} (tA_\lambda)^n \). Then in theory \( T_\lambda(t) \to T(t) \) as \( \lambda \to \infty \).

**Lemma 3.5.** Let \( A : \mathcal{D}(A) \to X \) be a linear operator with \( \mathcal{D}(A) \subseteq X \). Assume that (i) and (ii) of the Hille-Yosida theorem hold. Then, for all \( x \in X \), \( \lim_{\lambda \to \infty} \lambda R(\lambda;A)x = x \).
Proof. Let \( x \in \mathcal{D}(A) \) be given. For any \( \lambda > \omega \),
\[
(\lambda 1 - A)R(\lambda; A)x = x,
\]
\[
\lambda R(\lambda; A)x - x = AR(\lambda; A)x = R(\lambda; A)Ax,
\]
\[
\|\lambda R(\lambda; A)x - x\| = \|R(\lambda; A)Ax\| \\
\leq \frac{M}{\lambda - \omega}\|Ax\| \\
\to 0 \text{ as } \lambda \to \infty.
\]
Since \( \mathcal{D}(A) \) is dense in \( X \), the result follows. \( \square \)

Now we define the Yosida approximation \( A_{\lambda} \) of \( A \) for \( \lambda > \omega \). It is defined as
\[
A_{\lambda}x := \lambda AR(\lambda; A)x = (\lambda^2 R(\lambda; A) - \lambda 1)x.
\]
By Lemma 3.5, \( A_{\lambda}x \to Ax \) as \( \lambda \to \infty \) for all \( x \in \mathcal{D}(A) \).

**Lemma 3.6.** Let \( B \in \mathcal{L}(X; X) \) and define \( e^{tB} = \sum_{n=0}^{\infty} \frac{1}{n!}(tB)^n \) for all \( t \in \mathbb{R} \).

(i) \( \{e^{tB}\}_{t \geq 0} \) is a linear \( C_0 \)-semigroup with infinitesimal generator \( B \).

(ii) \( \lim_{t \to 0} \|e^{tB} - 1\| = 0 \).

(iii) For all \( \lambda \in \mathbb{K} \), \( e^{(tB-\lambda I)} = e^{-\lambda t}e^{tB} \).

**Proof.**

(i) Since \( B \in \mathcal{L}(X; X) \) we have \( \|B\| < +\infty \), and so for any \( x \in X \) and \( t \geq 0 \),
\[
\left\| \sum_{i=n}^{\infty} \frac{1}{i!}(tB)^i x \right\| \leq \sum_{i=n}^{\infty} \frac{(\|B\|)^i}{i!} \|x\| \leq \|x\|e^{\|B\|t},
\]

hence the sequence of partial sums \( \{\sum_{i=n}^{\infty} \frac{1}{i!}(tB)^i x\}_{n \in \mathbb{N}} \) is Cauchy in \( X \). Hence the series converges, \( e^{tB} \) is well defined, and \( e^{tB} \in \mathcal{L}(X; X) \).

Since for \( a, b \in \mathbb{R} \)
\[
\left( \sum_{n=0}^{\infty} \frac{1}{n!}a^n \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!}b^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!}(a + b)^n,
\]
we deduce that the semigroup property holds by the same argument which shows that \( e^{tB} \) is well defined. Clearly, \( \lim_{t \to 0} e^{tB} = e^{0B} = 1 \). Finally, to show that \( \{e^{tB}\}_{t \geq 0} \) is a linear \( C_0 \)-semigroup note that
\[
\|e^{tB}x - x\| = \left\| \sum_{n=1}^{\infty} \frac{1}{n!}(tB)^nx \right\| \leq \sum_{n=1}^{\infty} \frac{1}{n!}t^n \|B\|^n \|x\| = (e^{\|B\|t} - 1)\|x\| \to 0 \text{ as } t \downarrow 0.
\]

We claim that \( e^{\|B\|t} \) is differentiable with derivative \( Be^{tB} \). To see this note that
\[
B \int_0^t e^{sB} ds = B \sum_{n=0}^{\infty} \frac{1}{n!}(sB)^n ds = \sum_{n=0}^{\infty} B^{n+1} \int_0^t s^n ds = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(tB)^{n+1} = e^{tB} - 1.
\]

Now by differentiating both sides we deduce that
\[
\frac{d}{dt}e^{tB} = \lim_{h \to 0} \frac{e^{(t+h)B} - e^{tB}}{h} = Be^{tB}.
\]

Now since the infinitesimal generator is derivative at \( t = 0 \) we deduce that the infinitesimal generator of the linear \( C_0 \)-semigroup \( \{e^{tB}\}_{t \geq 0} \) is simply \( B \).

(ii) Since
\[
\|e^{tB} - 1\| = \left\| \sum_{n=1}^{\infty} \frac{1}{n!}(tB)^n \right\| \leq \sum_{n=1}^{\infty} \frac{1}{n!}t^n \|B\|^n = e^{\|B\|t} - 1,
\]
and \( e^{\|B\|/t} \to 1 \) as \( t \to 0 \), we deduce that \( \lim_{t \to 0} \|e^{tB} - 1\| = 0 \).
(iii) We have
\[ e^{(\beta - \lambda) \tau} = \sum_{n=0}^{\infty} \frac{1}{n!} (\tau B - \tau \lambda \mathbb{1})^n = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \tau B \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} (-\tau \lambda \mathbb{1})^n \right) = e^{\tau B} e^{-\lambda \tau \mathbb{1}}. \]

Now for any \( x \in X \) we see that
\[ e^{(\beta - \lambda) x} = e^{\tau B} e^{-\lambda x} = e^{\tau B} e^{-\lambda t} x = e^{-\lambda t} e^{\tau B} x. \]

In fact, it can be shown that if \( T \) is a linear \( C_0 \)-semigroup with the property that \( \lim_{t \to 0} ||T(h) - I|| = 0 \) then \( T(t) = e^{\tau B} \) for some \( B \in \mathcal{L}(X;X) \).

Now assume that conditions (i) and (ii) of the Hille-Yosida theorem hold, and let \( A_\lambda \) be the Yosida approximation of \( A \). Notice that for any \( \lambda > \omega \),
\[ e^{\tau A_\lambda} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{2n} t^n R(\lambda;A)^n}{n!} \]
\[ \|e^{\tau A_\lambda}\| \leq M e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{2n} t^n}{(\lambda - \omega)^n n!} \]
\[ = M e^{-\lambda t} \exp \left( \frac{\lambda^2}{\lambda - \omega} t \right) \]
\[ = M \exp \left( \frac{\lambda}{\lambda - \omega} t \right). \]

It follows that \( \|e^{\tau A_\lambda}\| \leq M e^{\omega_1 t} \) for any fixed \( \omega_1 > \omega \), for all \( \lambda \) sufficiently large when compared to \( \omega \).

Put \( T_\lambda(t) := e^{\tau A_\lambda} \) for all \( t \geq 0 \) and \( \lambda > \omega \). Notice that \( A_\lambda A_\mu = A_\mu A_\lambda \) and \( A_\lambda T_\mu(t) = T_\mu(t) A_\lambda \) for all \( \lambda, \mu > \omega \). Fix \( x \in \mathcal{D}(A) \).
\[ T_\lambda(t)x - T_\mu(t)x = \int_0^t \frac{d}{ds} \left( T_\mu(t-s)T_\lambda(s)x \right) ds \]
\[ = \int_0^t T_\mu(t-s)A_\lambda T_\mu(s)x - T_\mu(t-s)A_\mu T_\lambda(s)x \ ds \]
\[ = \int_0^t \left( T_\mu(t-s)T_\lambda(s) \right) (A_\lambda x - A_\mu x) \ ds. \]

So we deduce that
\[ \|T_\lambda(t)x - T_\mu(t)x\| \leq M^2 e^{\omega_1 t} ||A_\lambda x - A_\mu x||. \]

Hence \( \{T_\lambda(t)x\}_{\lambda > \omega} \) is uniformly Cauchy in \( t \) on bounded intervals. Since \( \mathcal{D}(A) \) is dense in \( X \) and since we have a bound on \( \|T_\lambda(t)\| \) (in \( \lambda \)), we have for all \( x \in X \), \( \lim_{\lambda \to \infty} T_\lambda(t)x \) exists.

For all \( t \geq 0 \) and \( x \in X \), put \( T(t)x = \lim_{\lambda \to \infty} T_\lambda(t)x \). Note that \( ||T(t)|| \leq M e^{\omega_1 t}, T(t)T(s) = T(t+s) \) for all \( s, t \geq 0 \), and \( T(0) = 1 \) – this follows since these relations all hold for each \( T_\lambda \). Continuity follows since the convergence is uniform on bounded intervals. Hence, we have shown that \( T \) is linear \( C_0 \)-semigroup. Let \( B \) be the infinitesimal generator of \( T \). Now we must show that \( B = A \). First we will show that \( B \) is an extension of \( A \), and then we will use a resolvent argument to show that \( \mathcal{D}(A) = \mathcal{D}(B) \). Let \( x \in \mathcal{D}(A) \) be given.
\[ ||T_\lambda(t)A_\lambda x - T(t)Ax|| \leq ||T_\lambda(t)(A_\lambda x - Ax)|| + \|((T_\lambda(t) - T(t))Ax)\| \]
\[ \leq M e^{\omega_1 t} ||A_\lambda x - Ax|| + ||(T - \lambda(t) - T(t))Ax|| \]
\[ \to 0 \text{ as } \lambda \to \infty. \]

Since the convergence is uniform in \( t \) on bounded intervals,
\[ T(t)x - x = \lim_{\lambda \to \infty} T_\lambda(t)x - x = \lim_{\lambda \to \infty} \int_0^t T_\lambda(s)A_\lambda x \ ds = \int_0^t T(s)Ax \ ds. \]
Now by the definition of $B$, for any $h > 0$,
\[
\frac{T(h)x - x}{h} = \frac{1}{h} \int_0^h T(s)Ax \, ds \to Ax \text{ as } h \downarrow 0.
\]

Hence, $x \in \mathcal{D}(B)$ and $Bx = Ax$. $B$ is closed since it is the infinitesimal generator of a linear $C_0$-semigroup, and $A$ is closed by assumption. Since $\|T(t)\| \leq Me^{\omega t}$ for any $\omega \geq \omega$, by Lemma 3.1, $\rho(B) \ni (\omega, \infty)$, so it follows that $\rho(B) \cap \rho(A) \neq \emptyset$. Choose $\lambda \in \rho(B) \cap \rho(A)$. By standard spectral theory since $A$ and $B$ are closed, $(\lambda - A)[\mathcal{D}(A)] = X$ and $(\lambda - B)[\mathcal{D}(A)] = X$. Furthermore, since $B$ extends $A$, $(\lambda - B)[\mathcal{D}(A)] = (\lambda - A)[\mathcal{D}(A)] = X$. To conclude the proof of the Hille-Yosida theorem, note that $\mathcal{D}(A) = R(\lambda; B)[X] = \mathcal{D}(B)$.

\[\square\]

Remark 3.7. Let $A : \mathcal{D}(A) \to X$ be a linear operator with $\mathcal{D}(A) \subseteq X$. The following are equivalent.

(i) $A$ is closed,
(ii) $(\lambda_1 - A) : \mathcal{D}(A) \to X$ is a bijection for some $\lambda \in \rho(A)$,
(iii) $(\lambda - A) : \mathcal{D}(A) \to X$ is a bijection for all $\lambda \in \rho(A)$.

\[\Diamond\]

Corollary 3.8. Assume that $A : \mathcal{D}(A) \to X$ is linear with $\mathcal{D}(A) \subseteq X$, and that $\mathcal{D}(A)$ is dense and $A$ is closed. Then $A$ generates a contractive linear $C_0$-semigroup if and only if $\rho(A) \ni (0, \infty)$ and $\|R(\lambda; A)\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

4. Contractive Semigroups

Let $T : [0, \infty) \to \mathcal{L}(X; X)$ be a contractive semigroup. For all $t, h \in [0, \infty)$,
\[
\|T(t + h)\| = \|T(t)T(h)\| \leq \|T(t)\| \|T(h)\| \leq \|T(t)\|,
\]
so $t \rightarrow \|T(t)\|$ is a decreasing function. Assume for now that $X$ is a Hilbert space. Let $x \in \mathcal{D}(A)$ be given, and put $\mu(t) = \|T(t)x\|^2 = (T(t)x, T(t)x)$. For all $t \geq 0$, since $\mu$ is decreasing,
\[
0 \geq \mu(t) = (T(t)x, T(t)Ax) + (T(t)Ax, T(t)x) = 2\Re(A\mu(t)x, x).
\]

In particular, for $t = 0$, $\Re(Ax, x) \leq 0$ for all $x \in \mathcal{D}(A)$.

We will prove that if $X$ is a Hilbert space and $A : \mathcal{D}(A) \to X$ is a linear operator then $A$ generates a contractive semigroup if and only if both of the following hold.

(i) $\Re(Ax, x) \leq 0$ for all $x \in \mathcal{D}(A)$, and
(ii) there exists $\lambda_0 > 0$ such that $\lambda_0A - A$ is surjective.

Definition 4.1. Let $X$ be a Banach space over $K$ with norm $\|\cdot\|$. A semi-inner product on $X$ is a mapping $\langle \cdot, \cdot \rangle : X \times X \to K$ such that

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$,
(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in X$ and $\alpha \in K$,
(iii) $\langle x, x \rangle = \|x\|^2$ for all $x \in X$, and
(iv) $\|x, y\| \leq \|x\|\|y\|$ for all $x, y \in X$.

\[\Diamond\]

Remark 4.2. The term "semi-inner product" is often used in a more general sense that is not linked to a pre-existing norm.

Now we ask: do semi-inner products exist, and can there be more than one associated with any given norm? The answer to both is yes in general. However, if $X^*$ is strictly convex then there cannot be more than one. We will see that if $\Re[Ax, x] \leq 0$ with respect to one semi-inner product then it holds with respect to any semi-inner product.

Proposition 4.3. There is at least one semi-inner product on a Banach space.
Lemma 4.7. Assume that \( A : \mathcal{D}(A) \to X \) is linear with \( \mathcal{D}(A) \subseteq X \). We say that \( A \) is dissipative if there is a semi-inner product on \( X \) such that \( \Re \langle Ax, x \rangle \leq 0 \) for all \( x \in \mathcal{D}(A) \).

The notion of dissipativity depends on the particular norm used, but it will turn out that it does not depend on the particular semi-inner product used.

Remark 4.5. Consider \( \mu_{\epsilon}(x, t) = \Delta \mu(x, t) - \alpha(x)\mu_t(x, t) \) with \( \mu \mid_{\partial\Omega} = 0 \), where \( \alpha \) is non-negative, smooth, with compact support, and \( \int_0^\infty \alpha > 0 \). Then solutions \( \mu \) tend to zero with \( t \).

Lemma 4.6. Assume that \( A : \mathcal{D}(A) \to X \) is linear with \( \mathcal{D}(A) \subseteq X \). Then \( A \) is dissipative if and only if \( \| (\lambda \mathbb{I} - A)x \| \geq \lambda \| x \| \) for all \( x \in \mathcal{D}(A) \) and \( \lambda > 0 \).

Proof. Assume that \( A \) is dissipative. Choose a semi-inner product such that \( \Re \langle Ax, x \rangle \leq 0 \) for all \( x \in \mathcal{D}(A) \). Then for all \( x \in \mathcal{D}(A) \) and \( \lambda > 0 \), we have

\[
\Re \langle (A - \lambda \mathbb{I})x, x \rangle = \lambda \| x \|^2 - \Re \langle Ax, x \rangle \geq \lambda \| x \|^2.
\]

Combining this with the fact that

\[
\Re \langle (\lambda \mathbb{I} - A)x, x \rangle \leq \| (\lambda \mathbb{I} - A)x \| \leq \| (\lambda \mathbb{I} - A)x \| \| x \|
\]

yields the result.

Assume now that \( \| (\lambda \mathbb{I} - A)x \| \geq \lambda \| x \| \) for all \( x \in \mathcal{D}(A) \) and \( \lambda > 0 \). As before, put

\[
\mathcal{F}(x) := \{ x^* \in X^* : \langle x^*, x \rangle = \| x \|^2 = \| x^* \|^2 \}.
\]

We identify three cases: \( x = 0 \), \( x \in \mathcal{D}(A) \setminus \{ 0 \} \) and \( x \notin \mathcal{D}(A) \).

Fix \( x \in \mathcal{D}(A) \setminus \{ 0 \} \). For all \( \lambda > 0 \) choose \( y_\lambda^* \in \mathcal{F}(\lambda x - Ax) \) and put \( z_\lambda^* = \frac{1}{\| y_\lambda^* \|} y_\lambda^* \).

\[
\lambda \| x \| \leq \| \lambda x - Ax \|
\]

by assumption

\[
= \frac{1}{\| y_\lambda^* \|} \langle y_\lambda^*, nx - Ax \rangle
\]

since \( y_\lambda^* \in \mathcal{F}(\lambda x - Ax) \)

\[
= \langle z_\lambda^*, \lambda x - Ax \rangle
\]

(this is a real number)

\[
= \Re \langle z_\lambda^*, Ax \rangle - \Re \langle z_\lambda^*, Ax \rangle.
\]

Since \( \| z_\lambda^* \| = 1 \) by construction,

\[
\lambda \| x \| \leq \Re \langle z_\lambda^*, Ax \rangle - \Re \langle z_\lambda^*, Ax \rangle \leq \lambda \| x \| - \Re \langle z_\lambda^*, Ax \rangle.
\]

Therefore, \( \Re \langle z_\lambda^*, Ax \rangle \leq 0 \) and similarly \( \Re \langle z_\lambda^*, x \rangle \geq \| x \| - \frac{1}{\lambda} \| Ax \| \). Since the unit ball in \( X^* \) is weak-* compact the net \( \{ z_\lambda^* \}_{\lambda \to \infty} \) has a weak-* cluster point \( z^* \in X^* \). Then \( \| z^* \| \leq 1 \), \( \Re \langle z^*, Ax \rangle \leq 0 \), and \( \Re \langle z^*, x \rangle \geq \| x \| \). It follows that \( \langle z^*, x \rangle = \| x \| \). Define a semi-inner product as before, but with

\[
F(x) = \begin{cases} 0 & x = 0 \\ \langle z^*, x \rangle & x \in \mathcal{D}(A) \setminus \{ 0 \} \\ \text{anything in } \mathcal{F}(x) & x \in X \setminus \mathcal{D}(A). \end{cases}
\]

Lemma 4.7. Assume that \( A : \mathcal{D}(A) \to X \) is linear with \( \mathcal{D}(A) \subseteq X \) and that \( A \) is dissipative. Let \( \lambda_0 \in (0, \infty) \) be given and assume that \( \lambda_0 \mathbb{I} - A \) is surjective. Then \( A \) is closed, \( \rho(A) \supseteq (0, \infty) \), and \( \| R(\lambda; A) \| \leq \frac{1}{\lambda} \) for all \( \lambda > 0 \).
Proof. Notice that, by Lemma 4.6, \(\|(\lambda I - A)x\| \geq \lambda\|x\|\) for all \(x \in \mathcal{D}(A)\) and \(\lambda > 0\). So we immediately deduce that \(\|R(\lambda; A)\| \leq \frac{1}{\lambda}\), provided the resolvent exists. The key points are to show that \(A\) is closed and that \(\lambda I - A\) is surjective for all \(\lambda > 0\).

Notice that \(\lambda_0 I - A\) is bijective since it is surjective and bounded below, and furthermore, \(\|(\lambda_0 I - A)^{-1}\| \leq \frac{1}{\lambda_0}\). So \((\lambda_0 I - A)^{-1} \in \mathcal{L}(X; X)\), hence it is closed, so \(A\) is closed as well.

To show that \(\rho(A) \supseteq (0, \infty)\) it suffices to show that \((A I - A)^{-1}\) is surjective for all \(\lambda > 0\). Let \(\Lambda = \{\lambda \in (0, \infty) : \lambda \in \rho(A)\}\), which is open (in the relative topology of \((0, \infty)\)) and non-empty. We will show that \(\Lambda\) is closed and conclude that \(\Lambda = (0, \infty)\). Let \(\{\lambda_n\}_{n=1}^{\infty}\) be a sequence in \(\Lambda\) converging to \(\lambda^* \in (0, \infty)\). We will show that \(\lambda^* I - A\) is surjective. Let \(y \in X\) be given. For every \(n \in \mathbb{N}\) let \(x_n = R(\lambda_n; A)y\). Note that \(\sup \{\frac{1}{\lambda_n} : n \in \mathbb{N}\} < \infty\).

\[
\|x_n - x_m\| = \|(R(\lambda_n; A) - R(\lambda_m; A))y\| \\
= |\lambda_n - \lambda_m|\|R(\lambda_n; A)R(\lambda_m; A)y\| \\
\leq |\lambda_n - \lambda_m| \frac{\|y\|}{\lambda_n \lambda_m} \\
\to 0 \text{ as } n, m \to \infty.
\]

So \(x_n \to x\) for some \(x \in X\). Finally, \(\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{D}(A), x_n \to x, \text{ and } Ax_n \to \lambda^* x - y\). Since \(A\) is closed, \((\lambda^* I - A)x = y\). □

**Theorem 4.8** (Lumer-Phillips, 1961). Assume \(A : \mathcal{D}(A) \to X\) is linear with \(\mathcal{D}(A)\) dense in \(X\).

(i) If \(A\) is dissipative and there is \(\lambda_0 > 0\) such that \(\lambda_0 I - A\) is surjective then \(A\) generates a contractive linear \(C_0\)-semigroup.

(ii) If \(A\) generates a contractive linear \(C_0\)-semigroup then \(\lambda I - A\) is surjective for all \(\lambda > 0\) and \(\mathfrak{M}[Ax, x] \leq 0\) for all \(x \in \mathcal{D}(A)\) and every semi-inner product on \(X\) (in particular, \(A\) is dissipative).

Proof. The first part follows from Lemma 4.7 and the Hille-Yosida theorem, since \(\|R(\lambda; A)\| \leq \frac{1}{\lambda}\) implies \(\|R(\lambda; A)^n\| \leq \frac{1}{\lambda^n}\).

For the second part, the surjectivity conclusion follows from the Hille-Yosida theorem. Let \([\cdot, \cdot]\) be a semi-inner product on \(X\). We need to show that \(\mathfrak{M}[Ax, x] \leq 0\) for all \(x \in \mathcal{D}(A)\). For all \(h > 0\) and \(x \in \mathcal{D}(A)\),

\[
\mathfrak{M}[T(h)x - x, x] = \mathfrak{M}[T(h)x, x] - \|x\|^2 \\
\leq \|T(h)x\|\|x\| - \|x\|^2 \\
\leq \|x\|^2 - \|x\|^2 \\
= 0.
\]

Dividing by \(h\) and letting \(h \downarrow 0\) yields \(\mathfrak{M}[Ax, x] \leq 0\). □

**Corollary 4.9.** Assume \(B : \mathcal{D}(B) \to X\) is linear with \(\mathcal{D}(B)\) dense in \(X\). Let \(\omega, \lambda_0 \in \mathbb{R}\) with \(\lambda_0 > \omega\) be given. If \(\lambda_0 I - B\) is surjective and there exists a semi-inner product on \(X\) such that \(\mathfrak{M}[Bx, x] \leq \omega\|x\|^2\) for all \(x \in \mathcal{D}(B)\), then \(B\) generates a linear \(C_0\)-semigroup \(T\) such that \(\|T(t)\| \leq e^{\omega t}\).

Proof. Let \(A = B - \omega I\) and apply the Lumer-Phillips theorem to \(A\). □

**Lemma 4.10.** Assume that \(X\) is reflexive and that \(A : \mathcal{D}(A) \to X\) is linear with \(\mathcal{D}(A) \subseteq X\). Let \(\lambda_0 > 0\) be given and assume that \(A\) is dissipative and that \(\lambda_0 I - A\) is surjective. Then \(\mathcal{D}(A)\) is dense in \(X\).

**Remark 4.11.** Let \(M\) be a linear submanifold in a Banach space \(X\) (not necessarily reflexive). Then \(M\) is dense in \(X\) if and only if for all \(y \in X\) there is a sequence \(\{x_n\}_{n=1}^{\infty} \subseteq M\) such that \(x_n \to y\) as \(n \to \infty\). Indeed, one direction is trivial. For the other, if \(y\) is not in the closure of \(M\) then \(\text{dist}(M, y) > 0\). By the Hahn-Banach theorem there is \(y^* \in X^*\) such that \(\langle y^*, x \rangle = 0\) for all \(x \in M\) and \(\langle y^*, y \rangle \neq 0\).
Proof. Let \( y \in X \) be given. It suffices to show that there is a sequence \( \{x_n\}_{n=1}^{\infty} \subseteq \mathcal{D}(A) \) such that \( x_n \to y \) as \( n \to \infty \).

Put \( x_n = \left(1 - \frac{1}{n}A\right)^{-1} y = nR(n;A)y \in \mathcal{D}(A) \) for all \( n \in \mathbb{N} \). Then

\[
\|x_n\| \leq n\|R(n;A)\|\|y\| \leq n\frac{1}{n}\|y\| = \|y\|.
\]

Choose a subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \) and \( x \in X \) such that \( x_{n_k} \to x \) as \( k \to \infty \). We are done if we show that \( x = y \). We have

\[
\begin{align*}
A \left( \frac{x_{n_k}}{n_k} \right) &= x_{n_k} - y \\
&\to x - y,
\end{align*}
\]

and \( x_{n_k} \to 0 \) (in fact, \( x_{n_k} \to 0 \)). Now since \( \text{Gr}(A) \) is closed and convex it is weakly closed. Since \( (0, x-y) \in \text{Gr}(A) \), we deduce that \( x = y \). \( \square \)

This lemma shows that if \( X \) is reflexive then we do not need to assume that \( \mathcal{D}(A) \) is dense in the Lumer-Phillips theorem. This is less helpful than it seems because in many applications it is trivial to check that the domain is dense.

**Theorem 4.12** (Lumer-Phillips for Hilbert spaces). Let \( X \) be a Hilbert space and assume that \( B : \mathcal{D}(B) \to X \) is linear with \( \mathcal{D}(B) \subseteq X \). Let \( \lambda_0, \omega \in \mathbb{R} \) and \( \lambda_0 > \omega \) be given. Assume that \( \Re(Bx, x) \leq \omega \|x\|^2 \) for all \( x \in \mathcal{D}(B) \) and that \( \lambda_0 I - B \) is surjective. Then \( B \) generates a linear \( C_0 \)-semigroup \( T \) such that \( \|T(t)\| \leq e^{\omega t} \) for all \( t \geq 0 \).

**Example 4.13.** Let \( \mathcal{D}(A) := \{u \in AC[0, 1] : u' \in AC[0, 1], u'' \in L^2[0, 1], u(0) = u(1) = 0\} \subseteq L^2[0, 1] \), and \( Au := u'' \). We have seen that \( A \) is closed and \( A \) is densely defined (in fact it is self-adjoint). For any \( u \in \mathcal{D}(A) \),

\[
(Au, u) = \int_0^1 u''u \, dx = -\int_0^1 (u')^2 \, dx \leq 0.
\]

If we can solve the ODE \( u'' = f \), \( u(0) = u(1) = 0 \) for any \( f \in L^2(0, 1) \), then \( A \) generates a contraction semigroup \( T \) by the Lumer-Phillips theorem. Thus the solutions to the heat equation

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 & \text{on } (0, 1) \\
u(t, 0) &= u(t, 1) = 0 & \text{for all } t \geq 0 \\
u(0, x) &= g(x) & \text{for all } x \in (0, 1)
\end{align*}
\]

can be written as \( u(x, t) = (T(t)g)(x) \). \( \diamond \)